Friends or Foes? The Interrelationship between Angel and Venture Capital Markets*

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Abstract

We examine the interrelationship between angel and venture capital markets, using a model with free entry and search frictions. Venture capitalists rely on the supply of angel-backed deals, and angels depend on competitive valuations from venture capitalists. Ex-post, venture capitalists have an incentive to hold-up angel investors, but ex-ante this may endanger their deal flow from angels. Endogenous entry limits the hold-up power of venture capitalists. In equilibrium, angels may encourage companies to exit early, in order to avoid venture capital funding altogether. The model also explains how hold-up and competition affect the valuations paid by angels and venture capitalists.

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1 Introduction

Angel investors are an important source of early stage funding for entrepreneurs. According to a report by the OECD, the size of the angel market is estimated to be roughly comparable to the venture capital market (OECD, 2011).[4] At the same time venture capital firms focus increasingly on later stage deals. As a result the funding path of growth-oriented start-ups typically involves some initial funding from angel investors, with subsequent funding coming from venture capitalists. Facebook and Google, arguably the most successful start-ups in recent history, both received angel financing prior to obtaining venture capital.

In the start-up ecosystem there is an interesting interdependence between angels and venture capitalists. On the one hand, angel investors have limited funds and typically need venture capitalists to provide follow-on funding for their companies. On the other hand, venture capitalists rely on angel investors for their deal flow. However, angel and venture capital investors have a fundamental conflict concerning valuation. Angel investors frequently complain that venture capitalists abuse their market power by offering unfairly low valuations. Expectations of low valuations at the venture capital stage then affect the willingness of angel investors to invest in early stage start-ups. Michael Zapata, an angel investor, explains it as follows (Holstein, 2012):

> "*In cases where the VCs do see a profit opportunity, they have become increasingly aggressive in low-balling the managements and investors of emerging companies by placing lower valuations on them.* [...] *Angels call these actions 'cram downs' or 'push downs'. The market has been very rough on the venture capitalists and they are making it tougher on the angels. They are killing their future deal flow by cramming them down, crashing them out.*"

In this paper we develop a theoretical framework for explaining the equilibrium interactions between angel and venture capital markets. We examine how the two markets are connected and explore what determines their respective market sizes. We endogenously derive the level of competition and the valuations that obtain in the two markets. We also examine how the efficiency of these two markets depends on company and market fundamentals.

Central to our model is the possibility that angels have limited funds, and that venture capitalists can hold them up when providing follow-on financing to their companies. One important insight is that the extent of hold-up critically depends on the level of competition in the venture capital market. A monopolist venture capitalist would have strong hold-up power over angels, but in a perfectly competitive venture capital market there would be no such hold-up power. We

[4] For 2009 the report estimates the US (European) venture capital market at $18.3B ($5.3B), and the US (European) angel market at $17.7B ($5.6B).

endogenize the level of competition using a search model à la Diamond-Mortensen-Pissaridis where despite free entry there is imperfect competition due to search frictions. This modeling approach naturally suits our context where entrepreneurs may spend considerable time searching for an appropriate investor match. In such a model venture capitalists may exploit some local monopoly power. However, if there are abnormal profits from doing so, more venture capitalists enter the market. In equilibrium we observe some local market power but no abnormal rents.[5]

A novel aspect of our model is that we consider the interrelationship between two connected search markets. To begin with, entrepreneurs search for angel investors. Search is costly and there is some urgency, since the entrepreneurs' opportunities can become obsolete. If an entrepreneur finds an angel, the two agree on a deal, and the angel provides funding for the initial development phase. If this phase is successful, the company has a growth option that requires further funding. We assume that angels do not have enough capital to finance the growth option, and therefore rely on the venture capital market. Searching for venture capitalists is costly. Once a venture capitalist is found, a new deal has to be structured. Renegotiation of the original contract is always possible, so the main protection angel investors have is their right to refuse a deal. We use the Shapley value to determine the outcome of the tripartite negotiation between the entrepreneur, angel, and venture capitalist. The outside options of the entrepreneur and angel are determined by the level of competition, specifically the expected time and cost of finding an alternative venture capitalist. Once a deal is struck with a venture capitalist, the company develops its products and generates some expected return. We derive the steady-state equilibrium of these two interconnected markets.

Within each market the model generates intuitive comparative statics results. For example, higher investment requirements and higher investor search costs, decrease the size of the market and reduce its competitiveness (as measured by the ratio of investors to companies in the market; this is also known as market density in the search literature). This naturally leads to a lower deal rate (as measured by the percentage of new ventures that receive funding). Remarkably, we find the opposite when entrepreneurs face higher search costs or when their ideas become obsolete at a higher rate. A weaker bargaining position for entrepreneurs attracts more investors, so that the markets become bigger and more competitive, with higher deal rates and higher valuations.

The most novel results pertain to cross-market effects. Effects in the venture capital market can trickle down into the angel market. For example, higher investment requirements and higher investor search costs in the venture capital market result in smaller angel markets, less competi-

[5]Our equilibrium concept is broadly consistent with the empirical evidence on venture capital markets. Even though venture capital markets are not perfectly competitive, venture capital firms do not earn abnormal returns. See Da Rin, Hellmann and Puri (2013) for an overview of the venture capital literature, and Robinson and Sensoy (2013) for the most recent evidence on venture capital returns.

tion among angels, lower deal rates, and lower valuations for angel rounds. Interestingly, higher company search costs at the venture capital stage have opposite effects across markets. Within the venture capital market they encourage entry, and thereby raise the level of competition. In the angel market, however, higher company search costs discourage angel investors, resulting in a smaller and less competitive angel market, lower deal rates and lower valuations. The same applies to the rate at which entrepreneurial opportunities become obsolete at the venture capital stage: a higher rate leads to a larger and more competitive venture capital market, but to a smaller and less competitive angel market.

Concerning the effect of angel markets on venture capital, we find that the structure of the angel market affects only the size of the venture capital market, but not its competitive structure, the deal rate, or the equilibrium valuations paid. Specifically we find that higher investment costs in the angel market reduce the size of the venture capital market. Similarly, higher search costs for angel investors also reduce the equilibrium deal flow to venture capitalists. The key intuition is that characteristics of the angel market affect the flow of new deals that come into the venture capital market, but they do not affect the competitive structure of that market, which is entirely driven by the venture capitalists' entry and search conditions.

Central to our analysis is the hold-up problem for angels at the venture capital stage. We introduce a parameter that naturally measures the extent to which angel investors are vulnerable to hold-up. We find that the more vulnerable angels are, the smaller the size of the angel market, and the lower its degree of competition. The effects on the venture capital market are more complex: on the one hand the bigger rents from hold-up encourage entry, on the other the suppression of angels results in a smaller deal flow. While the effect of angel vulnerability on the size of the venture capital market is ambiguous, the effect on competition is unambiguously positive, since there are more venture capitalists looking for fewer deals.

In our base model we assume that there are no binding constraints on the initial deal between the entrepreneur and the angel investor. This assumption holds as long the relative contributions, namely the labor provided by the entrepreneur, and the capital provided by the angel investor, are not too dissimilar. As a model extension we consider the case where the capital contribution of the angel investor is significantly larger than the labor contribution of the entrepreneurs. Under these circumstances it may be impossible for the two parties to reach an 'equitable deal', as defined by the Nash bargaining solution with transferable utility. We show that such a constraint reduces the size of the angel market, dampens competition, and lowers the deal rate.

Our paper draws on several theory literatures. A growing number of papers examines the implications of staged financing arrangements. Neher (1999) and Bergemann, Hege, and Peng (2009) study the design of optimal investment stages. Admati and Pfleiderer (1994) and Fluck, Garrison, and Myers (2005) consider the differential investment incentives of insiders and outsiders at the refinancing stage. Building on recent work about tolerance for failure (Manso, 2011), as well as the literature on soft budget constraints (Dewatripont and Maskin, 1995), Nanda and Rhodes-Kropf (2012a) consider how investors optimally choose their level of failure tolerance in a staged financing model. These models all assume that the original investors can finance the additional round. Our model departs from this assumption by focusing on smaller angel investors who do not have the financial capacity to provide follow-on financing.

The theory closest to ours is the recent work by Nanda and Rhodes-Kropf (2012b) on financing risk. They too assume that the initial investors cannot provide all the follow-on financing. Their analysis focuses on the possibility of multiple equilibria in the late stage market, and shows how different expectations about the risk of refinancing affect initial project choices. Our model does not focus on financing risk, but instead focuses on the hold-up problem at the refinancing stage.

In a seminal paper, Inderst and Müller (2004) first introduce search into a model of entrepreneurs and investors. They focus on how competitive dynamics affect venture capital valuations. Silveira and Wright (2006) and Nanda and Rhodes-Kropf (2012a) also use similar model specifications for other purposes. One limitation of all these search models (including ours) is that they require homogenous types. Hong, Serfes, and Thiele (2013) consider a single-stage venture capital financing model with matching among heterogeneous types.

Our model distinguishes between angels and venture capitalists on the basis of the investment stage and the amount of available funding: angels can only invest early and have limited funds, venture capitalists can only invest later and have sufficient funds to do so.[6] While we focus on some important distinctions between angels and venture capitalists, our model clearly doesn't capture all the nuances of reality. First, it is sometimes difficult to draw a precise boundary between what constitutes an angel investor versus a venture capitalist. Shane (2008) and the OECD report (OECD, 2011) provide detailed descriptions of angel investing, and the diversity within the angel community. Second, we do not model value-adding activities of angels versus venture capitalists. Chemmanur and Chen (2006) assume that only venture capitalists but not angels can provide value-added services. By contrast, Schwienbacher (2009) argues that both angels and venture capitalists may provide such services, but that angel investors provide more effort because they still need to attract outside investors at a refinancing stage.[7] Third, in our model both angels and venture capitalists are pure profit maximizers. The work of Axelson, Strömberg, and Weisbach (2009) suggests that the behavior of venture capitalists may

[6]The empirical evidence in Goldfarb et al. (2012) and Hellmann, Schure and Vo (2013) is broadly supportive of these assumptions.

[7]In terms of related empirics, Hellmann and Puri (2002) provide evidence on the value-adding activities of venture capital versus angel investors. Kerr, Lerner and Schoar (2013) examine the effects that angel groups have on company performance.

be influenced by agency considerations. Moreover, angel investors can be motivated by non-financial considerations, such as personal relationships or social causes, as discussed in the work of Shane (2008) and Van Osnabrugge and Robinson (2000). Finally, while we motivate our paper with angels and venture capitalists, our theory applies more broadly to the relationship between early and late investors. Further examples of early investors include friends and family, accelerators, and other specialized early stage investors, such as university-based seed funds and others. Further examples of late investors include corporate investors, and a variety of other financial institutions such as banks, growth capital funds, or family offices.

Our paper is also related to the literature on the staged commercialization of new venture ideas. Teece (1986), Anand and Galetovics (2000), Gans and Stern (2000) and Hellmann and Perotti (2011) all consider models where complementary asset holders have a hold-up opportunity at a later stage. They mainly ask how this hold-up problem impacts the optimal organization of the early stage development efforts. This paper focuses on the challenges of financing ventures across such different commercialization stages.

The remainder of the paper is structured as follow. Section 2 introduces our main model. Section 3 derives and analyzes the stationary equilibria of the angel and venture capital markets. Section 4 analyzes the interrelationship between the angel market and the venture capital market. Section 5 analyzes how the hold-up of angel investors affects the angel market as well as the venture capital market. Section 6 examines the decision of angels to exit early, in order to avoid the venture capital market altogether. In Section 7 we analyze the challenges of funding projects that are particularly capital-intense. Section 8 summarizes our main results and concludes. All proofs are in the Appendix.

2 The Base Model

Our analysis focuses on endogenously deriving the size and competitive structure of the early and late stage markets. Conceptually we want a model with endogenous entry, and a model with a meaningful continuum between monopoly and perfect competition. This naturally leads us to a search model in the style of Diamond, Mortensen and Pissaridis.[8] This type of model allows for free entry, and it endogenously generates the equilibrium market density, which is a natural and continuous measure of competition. Moreover, the real-life search process of entrepreneurs looking for investors closely resembles the assumptions of pairwise matching used in the theoretical search models (see Inderst and Müller (2004)).

[8]See Pissarides (1979, 2000), Mortensen and Pissarides (1994), and Diamond (1982, 1984).

We consider a continuous time model with three different types of risk-neutral economic agents: entrepreneurs, angel investors, and venture capitalists (VCs). The length of one period is $\Delta \to 0$, and the common discount rate is $r > 0$. In each period an exogenous number of wealth-constrained entrepreneurs, $\overline{m}_1^E \geq 1$, seek external financing for their start-up companies.[9] Each entrepreneur needs two financing stages to successfully establish his new venture; see 1 for a graphical overview.

Our model distinguishes between an *early stage* and a *late stage* market. Specifically, each entrepreneur needs an early stage investment k_1, and a late stage investment k_2, with $k_2 > k_1$. We assume that early and late stage financing is provided by two distinct types of investors. For clarity of exposition we associate angels with early stage investments, and VCs with late stage investments. In reality VCs may sometimes also invest in early stage deals, and angels in late stage deals. This does not affect the basic insights of the model; all that matters is that there is some separation of early and late stage markets, where a company that obtained funding from an investor in the early stage needs to find a new investor at the late stage. This may be either because the early stage investor does not have the funds to fully finance the late stage investment, or because early and late stage investors have different skills and information, making both of them essential to the success of the venture. For simplicity we also assume that early stage investors do not contribute at all to the late stage investment. Allowing them to finance part of the later stage investment would not change anything.[10]

In the early stage each entrepreneur needs to find an angel investor, who can make the required investment k_1. We assume a monopolistically competitive search market with free entry. Specifically, in each period m_1^A angel investors enter the early stage market seeking investment opportunities, where m_1^A is endogenous and satisfies the zero-profit condition for angels. We denote M_1^A as the equilibrium stock of angels in the early stage market at a given point in time, and M_1^E as the equilibrium stock of entrepreneurs.

Entrepreneurial opportunities often depend on speedy execution, so that delays in fundraising can be costly for the entrepreneurs. We therefore augment the standard search model with a parameter that measures urgency, or the cost of delay. Specifically we assume that a fraction $\delta_1 M_1^E$ of business ideas become obsolete in each period, generating a zero payoff. We refer to

[9]Endogenizing the number of entrepreneurs – in addition to the number of investors – would lead to multiple equilibria in the search market. And because our main focus is on endogenous entry of investors, we keep the number of entrepreneurs fixed.

[10]All that matters for the model is that angels do not have enough wealth to finance both stages. Conversely we also assume that VCs cannot be the sole investor in both the early and late stage market. Our assumptions closely matches industry behavior where even early stage VCs typically seek syndication partners for the later investment stages.

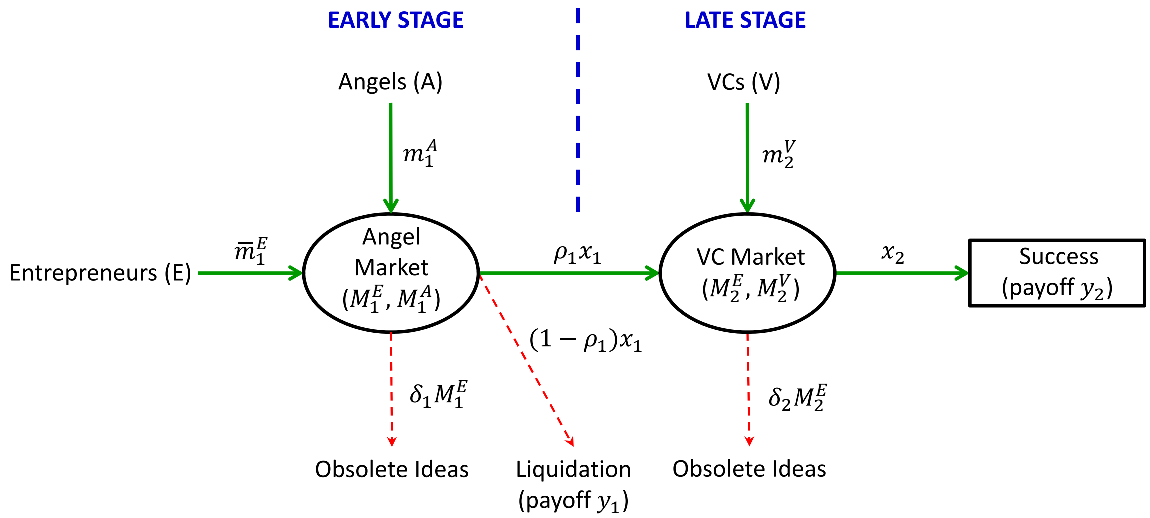

Figure 1: Financing Stages – Angel and VC Markets

δ_1 as the *death rate* of early stage ventures, which reflects the urgency for start-up companies to receive angel investments.[11]

Our model includes the standard search model parameters. We denote the individual search cost for entrepreneurs in the early stage market by σ_1^E, and the search cost for angels by σ_1^A.[12] The expected utilities from search in the early stage market is U_1^E for entrepreneurs, and U_1^A for angels. We use a standard Cobb-Douglas matching function $x_1 = \phi_1 \left[M_1^A M_1^E\right]^{0.5}$ with constant returns to scale, where x_1 is the number of entrepreneur-angel matches in each period, and $\phi_1 > 0$ is an efficiency measure of the matching technology in the early stage market. Once matched, the angel and entrepreneur bargain over the allocation of equity. We use the symmetric Nash bargaining solution to derive the equilibrium outcome of this bilateral bargaining game. The angel then invests k_1 in the new venture, and the entrepreneur provides labor at the personal cost l_1.

A distinct feature of our model is the assumption of staged financing, and the existence of two interrelated markets. An early stage venture that received the angel investment k_1 has a growth opportunity with probability ρ_1. In that case it becomes attractive enough for VCs to make a follow-on investment k_2. Without a growth opportunity the early stage venture can only be liquidated, generating the payoff y_1, where y_1 is a random variable with support $[0, \overline{y}_1]$. For now we assume that $\overline{y}_1$ is sufficiently low, so that it is always optimal for ventures with growth

[11]Our model does not rely on a cost of delay, in that all the results continue to hold for $\delta_1 = 0$. However, allowing for $\delta_1 > 0$ generates new insights, in particular concerning the deal rate, discussed below.

[12]Naturally we focus on the case where the angel market exists, which requires that σ_1^A is not too large.

opportunities to seek follow-on investments from VCs.[13] Thus, in each period the number of ventures moving on to the late stage market is given by $\rho_1 x_1$.

The owners of each of the start-ups with growth opportunities – namely the entrepreneur and the angel investor – search for a VC in the late stage market. Naturally the angel investment k_1 is sunk at the late stage. Moreover, as discussed above, we make an innocuous simplifying assumption that the early stage investors do not contribute to the late stage investment, so that all of k_2 is provided by VCs. Again we assume that the late stage market is monopolistically competitive with free entry. We denote m_2^V as the endogenous number of VCs entering the market in each period. The equilibrium stock of VCs in the late stage market at a given point in time is M_2^V, and the equilibrium stock of ventures requiring VC financing is M_2^E. As for the angel market, we assume that a fraction $\delta_2 M_2^E$ of business ideas becomes obsolete in each period, generating a zero payoff. The individual search cost in the late stage market is σ_2^i, and the expected utility from search is U_2^i, with $i = E, A, V$.[14] For tractability we assume that entrepreneurs and angels – as joint owners of late stage start-ups – incur the same cost when searching for a VC investor, i.e., $\sigma_2 \equiv \sigma_2^E = \sigma_2^A$. The matching function for the late stage market is $x_2 = \phi_2 \left[M_2^V M_2^E\right]^{0.5}$, where x_2 is the number of start-ups that receive VC financing in each period, and $\phi_2 > 0$ is a measure of the matching efficiency. Once the owner of a start-up (entrepreneur and angel) found a VC investor, they all bargain over the allocation of equity, as discussed below. Once the parties reach an agreement the VC makes the required follow-on investment k_2, and the entrepreneur provides additional labor at the personal cost l_2. The venture then generates the payoff $y_2 > 0$.[15]

In the later stage market the bargaining game is between three key players. We use the Shapley value to derive the outcome of this trilateral bargaining game.[16] This implicitly assumes that all prior contracts can be renegotiated. While the entrepreneur and angel may want to commit to a specific equity allocation beforehand, the VC never agreed to that, and may therefore ask that all prior arrangements be ignored. In equilibrium the three parties agree on a division of shares that is determined by their outside options looking forward. Because renegotiation is fully anticipated ex-ante, lack of commitment turns out not to be a very restrictive assumption. However, in Section 7 we discuss some extreme parameter constellations, such as very high

[13]In Section 6 we relax this assumption, and allow entrepreneurs and angels to choose an early exit despite growth opportunities.

[14]To ensure existence of the VC market, we assume that σ_2^V is not too large.

[15]Note that y_2 can be interpreted as a constant, or as the expected value of a random return.

[16]The Shapley value is widely regarded as the most natural extension of the Nash Bargaining solution to games with more than two players. It provides an intuitive allocation of equity, which reflects the marginal contribution of each party to the value generation.

capital intensity, where a commitment between the entrepreneur and angel not to renegotiate their stakes could in principle affect the equilibrium.

Given the simple payoff structure of the model, equity is always an optimal security. The angel investor receives an equity stake α in the early stage (which is likely to be renegotiated later), and the VC receives an equity stake β^V. The so-called postmoney valuation is then given by $V_1 = k_1/\alpha$ for angel rounds, and $V_2 = k_2/\beta^V$ for VC rounds.[17]

3 Market Equilibria

3.1 Angel Market Equilibrium

We first establish and analyze the equilibrium of the angel market. For this we take the expected utilities from search in the VC market, U_2^A and U_2^E, as given (we derive U_2^A and U_2^E later in Section 3.2).

We start by looking at how entrepreneurs and angels split the expected surplus when making a deal. Our model consists of two investment stages, so the early stage allocation of surplus naturally depends on the partners' expected utilities from search in the late stage market, U_2^E and U_2^A. However, as discussed above, the entrepreneur and angel cannot affect these utilities, because of an inability to commit not to renegotiate. The early stage equity allocation therefore only determines how the liquidation value y_1 is split, with the angel receiving αy_1 and the entrepreneur keeping $(1-\alpha)y_1$.

We call the utilities at the time of making a deal the *deal values*, and denote them by D_1^E and D_1^A. In the angel market they are given by

$$\begin{aligned} D_1^E &= \rho_1 U_2^E + (1-\rho_1)(1-\alpha)y_1 - l_1 \\ D_1^A &= \rho_1 U_2^A + (1-\rho_1)\alpha y_1 - k_1. \end{aligned}$$

The optimal equity share for the angel, α^*, satisfies the symmetric Nash bargaining solution, which accounts for the outside option of each party. The outside option for the entrepreneur is

[17] We also note that while our model set-up is suitable for analyzing valuations, it does not generate any deep new insights into financial security structures. Angels and VCs sometimes use more elaborate securities, such as preferred equity or convertible notes (see Kaplan and Strömberg (2003) and Hellmann (2006)). In this model none of these more complicated securities can achieve anything better than simple equity. This is because in case of VC financing, the angel's choice of security is irrelevant – the original contract will get renegotiated anyway. Technically we note that none of the coalitional values of the Shapley game depend on the securities held by the angel investor. Moreover, in case of liquidation, there is an exogenous liquidation value, rendering security structure unimportant. We will briefly return to the discussion of optimal securities in Section 7.

to search for a new angel investor, which would give him the expected utility U_1^E. The angel could also search for a new entrepreneur; however, free entry implies $U_1^A = 0$. We define $\widetilde{D}_1^E$ and $\widetilde{D}_1^A$ as the deal values reflecting the Nash bargaining solution, which are given by

$$\begin{aligned}\widetilde{D}_1^E &= \frac{1}{2}\left[\rho_1\left(U_2^E + U_2^A\right) + (1-\rho_1)y_1 - k_1 - l_1 + U_1^E\right] \\ \widetilde{D}_1^A &= \frac{1}{2}\left[\rho_1\left(U_2^E + U_2^A\right) + (1-\rho_1)y_1 - k_1 - l_1 - U_1^E\right].\end{aligned}$$

The entrepreneur gets half of the expected surplus from the deal, plus half the value of his outside option U_1^E. The angel receives the remaining surplus. The equilibrium equity for the angel, α^*, then ensures that the actual deal values (D_1^E and D_1^A) reflect the Nash bargaining solution ($\widetilde{D}_1^E$ and $\widetilde{D}_1^A$). To derive α^*, we first note that the angel's and entrepreneur's expected utilities from search in the VC market are symmetric, i.e., $U_2^A = U_2^E$. We will formally show this in Section 3.2. Solving either $D_1^E = \widetilde{D}_1^E$ or $D_1^A = \widetilde{D}_1^A$ for α then yields

$$\alpha^* = \frac{1}{2} + \frac{k_1 - l_1 - U_1^E}{2(1-\rho_1)y_1}. \tag{1}$$

Depending on the values of k_1 and l_1 (as well as the outside option U_1^E), angels receive more or less than half of the equity. Importantly, (1) also implies that we need to distinguish between two main scenarios: If l_1 and k_1 are not too dissimilar, then the equilibrium equity share α^* satisfies $\alpha^* \in [0,1]$. We call this the *unconstrained case*. Otherwise we have $\alpha^* \notin [0,1]$, so that either the angel would have to make a transfer payment to the entrepreneur ($\alpha^* < 0$), or the entrepreneur would have to make a payment to the angel ($\alpha^* > 1$). We call this the *constrained case*. For our main analysis we focus on the unconstrained case. In Section 7 we consider the constrained bargaining case.

Given the deal values $\widetilde{D}_1^E$ and $\widetilde{D}_1^A$, we can now characterize the dynamic equilibrium of the angel market. Let $q_1^E \equiv x_1/M_1^E$ denote the deal arrival rate for entrepreneurs, which measures the probability than an entrepreneur finds an angel investor at a given point in time. The expected utility of an entrepreneur when searching for an angel investor, U_1^E, is then defined by the following asset equation:

$$rU_1^E = q_1^E(\widetilde{D}_1^E - U_1^E) + \delta_1(0 - U_1^E) - \sigma_1^E.$$

This equation says that the discounted expected utility from search equals the expected value of getting a deal ($q_1^E(D_1^E - U_1^E)$), minus the expected costs when searching for an angel investor:

the risk of the business idea becoming obsolete ($\delta_1 U_1^E$) and the direct cost of search (σ_1^E). Solving the asset equation we find the entrepreneur's expected utility from search:

$$U_1^E = \frac{-\sigma_1^E + q_1^E \widetilde{D}_1^E}{r + \delta_1 + q_1^E}. \tag{2}$$

Likewise we can find the angel's expected utility from search:

$$U_1^A = \frac{-\sigma_1^A + q_1^A \widetilde{D}_1^A}{r + q_1^A},$$

where $q_1^A \equiv x_1/M_1^A$ is the deal arrival rate for angels. Because of free entry, the expected utility from search for angels must be zero in equilibrium. Thus, in equilibrium we have

$$q_1^A \widetilde{D}_1^A = \sigma_1^A. \tag{3}$$

For now we assume that the search cost for angels σ_1^A is not too large, so that some angels will always find it optimal to enter the market. Later we provide a precise condition for the existence of the angel market.

In a stationary equilibrium the stock of entrepreneurs must be constant. This requires that the total outflow of entrepreneurs – because either their business ideas became obsolete ($\delta_1 M_1^E$) or they found an angel investor ($q_1^E M_1^E$) – is equal to the (exogenous) inflow of entrepreneurs ($\overline{m}_1^E$):

$$\delta_1 M_1^E + q_1^E M_1^E = \overline{m}_1^E. \tag{4}$$

Likewise, in a stationary equilibrium the outflow of angels ($q_1^A M_1^A$) is equal to the (endogenous) inflow of angels (m_1^A):

$$q_1^A M_1^A = m_1^A. \tag{5}$$

The equilibrium of the angel market is defined by conditions (2), (3), (4), and (5).

We are interested in four key characteristics of the angel market equilibrium. First, we want to understand the determinants of the absolute size of the angel market. This can be measured by the equilibrium stock of angel investors M_1^A at a given point in time. Second, we want to understand the determinants of the degree of competition among angel investors. In a search model this can be readily measured by the number of angels relative to the number of entrepreneurs in the market, also known as market thickness. Formally we denote the degree of angel market competition by θ_1, where $\theta_1 = M_1^A/M_1^E$. Third, as a measure of market efficiency, we are interested in the percentage of ventures that get funded in the early stage market. Formally we denote the early stage deal rate by ξ_1, with $\xi_1 = x_1/\overline{m}_1^E$. Our assumption of a positive death rate ($\delta_1 > 0$) allows for the possibility of $\xi_1 < 1$, so that the model generates some interesting comparative statics results. Fourth, our model generates some novel predictions about market valuations. We focus on the so-called post-money valuation given by $V_1 \equiv k_1/\alpha$.

The next proposition provides a comprehensive summary of the comparative statics results for the angel market.

Proposition 1 *Consider the angel market.*

(i) *The equilibrium size of the angel market M_1^{A*} is increasing in σ_1^E, and decreasing in l_1, k_1 and σ_1^A.*

(ii) *The equilibrium degree of competition θ_1^* is increasing in ϕ_1, σ_1^E and δ_1, and decreasing in l_1, k_1 and σ_1^A.*

(iii) *The equilibrium deal rate ξ_1^* is increasing in ϕ_1 and σ_1^E, and decreasing in δ_1, l_1, k_1, and σ_1^A.*

(iv) *The equilibrium valuation of early stage start-up companies V_1^* is increasing in ϕ_1, l_1 and k_1, and decreasing in σ_1^E, σ_1^A and δ_1.*

Our model generates intuitive comparative statics. Higher direct costs for angel investors (higher k_1 and σ_1^A) lead to less entry, so that in equilibrium fewer angels search investment opportunities. This implies a smaller and less competitive angel market (lower M_1^{A*} and θ_1^*). In contrast, higher direct and indirect costs of search for entrepreneurs (higher σ_1^E and δ_1) weaken their bargaining position, so that angels get better deals when investing in early stage start-ups. This encourages more entry of angels, and therefore leads to a more competitive angel market (higher θ_1^*).[18] A higher required labor input from the entrepreneur (l_1) naturally lowers the deal value for angels, thus discouraging entry. This makes the angel market smaller and less competitive.

Concerning market efficiency, in a more competitive angel market all entrepreneurs are more likely to find an investor. This explains why the comparative statics results for the degree of competition θ_1^* and for the deal rate ξ_1^*, are essentially the same. The only exception is the early stage death rate δ_1, which has a negative effect on the percentage of deals ξ_1^* – as one would expect.

[18]While a higher σ_1^E also implies a bigger angel market (higher M_1^{A*}), the effect of δ_1 on M_1^{A*} is ambiguous. This is because a higher δ_1 has two opposite effects: First, ceteris paribus, it reduces the equilibrium stock of entrepreneurs. Because of fewer investment opportunities, a smaller number of angels then enters the market. Second, a higher δ_1 weakens the bargaining position for entrepreneurs, so that angels get better deals. This in turn encourages entry of more angels. Which effect dominates depends on the specific parameter values.

To understand our valuation results it is useful to remember that a higher equity share α^* for angels implies lower post-money valuations V_1^*. Lower search costs for angels (lower σ_1^A) and higher direct and indirect search costs for entrepreneurs (higher σ_1^E and δ_1) all weaken the outside option U_1^E for entrepreneurs, and therefore lower valuations. A higher required labor input l_1 reduces the equilibrium equity share α^* for angels, and therefore results in higher early stage valuations. Moreover, we also find that a higher k_1 leads to higher post-money valuations of start-ups. Even though angels then require a higher equity share α^* – which would ceteris paribus reduce post-money valuations – we find that the direct effect of a higher k_1 always dominates. The equilibrium valuation is thus increasing in the required early stage investment k_1.

3.2 VC Market Equilibrium

We now characterize the equilibrium of the VC market. The inflow of new ventures into the VC market, $\rho_1 x_1^*$, is defined by the angel market equilibrium. We fix $\rho_1 x_1^*$ for now, and focus exclusively on the within-market effects. In Section 4 we then connect the angel and the VC markets, and analyze the cross-markets effects.

In the late stage market the owners of each start-up company – the entrepreneur and angel – seek an investment k_2 from a VC. Given that the angel investment k_1 is sunk, the total surplus, which we denote by π, is defined by $\pi = y_2 - k_2 - l_2$. We use the Shapley value to derive the outcome of this tripartite bargaining game. The outside option for the entrepreneur and angel is to go back to the market and search for a new VC investor; the joint value of their outside option is therefore given by $U_2^E + U_2^A$. The VC can also search for a new investment opportunity; however, free entry implies $U_2^V = 0$. For now we assume that the success of a late stage start-up requires the involvement of all three parties, i.e., it is impossible to exclude any one of the three partners. In Section 5 we consider a model extension where the entrepreneur and VC can establish a new venture based on the same business idea, thereby excluding the angel investor.

In the Appendix we derive the following late stage deal values for the entrepreneur (D_2^E), angel (D_2^A), and VC (D_2^V), using the Shapley value:

$$D_2^E = \tfrac{1}{3}\pi + \tfrac{1}{6}\left[U_2^E + U_2^A\right], \quad D_2^A = \tfrac{1}{3}\pi + \tfrac{1}{6}\left[U_2^E + U_2^A\right], \quad D_2^V = \tfrac{1}{3}\pi - \tfrac{1}{3}\left[U_2^E + U_2^A\right]$$

Each entrepreneur and angel gets one-third of the total surplus π, plus a premium which reflects the joint value of their outside option ($U_2^E + U_2^A$). The VCs receive the remaining surplus. The equilibrium deal values define the late stage equity shares that all parties agree on, which we denote by β^i, $i = E, A, V$. For parsimony we state the equilibrium equity shares in the Appendix. We can then use the late stage deal values D_2^i, $i = E, A, V$, to characterize the stationary equilibrium of the VC market. Recall that entrepreneurs and angels – as co-owners of a late stage start-up company – have identical costs when searching for a VC investor, with $\sigma = \sigma_2^E = \sigma_2^A$. Moreover, their deal values D_2^E and D_2^A are identical. This implies that their expected utilities from search in the late stage market must be the same in equilibrium, i.e., $U_2^E = U_2^A$.[19] Let $q_2^E \equiv x_2/M_2^E$ denote the deal arrival rate for entrepreneur-angel pairs, and $q_2^V \equiv x_2/M_2^V$ the deal arrival rate for VCs. The stationary VC market equilibrium is then defined by the following four conditions:

$$U_2^A = U_2^E = \frac{-\sigma_2 + q_2^E D_2^E}{r + \delta_2 + q_2^E} \tag{6}$$

$$q_2^V D_2^V = \sigma_2^V \tag{7}$$

$$\delta_2 M_2^E + q_2^E M_2^E = \rho_1 x_1^* \tag{8}$$

$$q_2^V M_2^V = m_2^V. \tag{9}$$

The expected utility for entrepreneurs and angels from search in the late stage market is given by (6). Condition (7) is the free-entry condition for VCs, which implies that $U_2^V = 0$ in equilibrium. Condition (8) ensures that the total outflow of new ventures equals the total inflow. Likewise, condition (9) guarantees that the outflow of VCs equals the endogenous inflow.

Again we focus on four main variables of interest: (i) the absolute size of the VC market M_2^V, (ii) the degree of competition among VCs $\theta_2 = M_2^V/M_2^E$, (iii) the deal rate in the late stage market, $\xi_2 = x_2/(\rho_1 x_1^*)$, and (iv) the valuation of late stage start-ups $V_2 = k_2/\beta^V$. The next proposition provides an extensive summary of the comparative statics results, focusing on the parameters that are associated with the VC market.[20]

Proposition 2 *Consider the VC market.*

- (i) *The equilibrium size of the VC market M_2^{V*} is increasing in σ_2, and decreasing in l_2, k_2 and σ_2^V.*

- (ii) *The equilibrium degree of competition θ_2^* is increasing in ϕ_2, σ_2 and δ_2, and decreasing in l_2, k_2 and σ_2^V.*

[19] This condition greatly simplifies many of our basic comparative statics. However, in Section 5 we consider a model extension where the late stage expected utilities of angels and entrepreneurs differ.

[20] It is important here to keep in mind that the inflow of new ventures into the VC market is exogenous for now. In Section 4 we briefly discuss whether and how the results change when connecting both search markets, so that the inflow into the VC market is endogenous.

(*iii*) *The equilibrium late stage deal rate* ξ_2^* *is increasing in* ϕ_2 *and* σ_2*, and decreasing in* δ_2*,* l_2*,* k_2*, and* σ_2^V*.*

(*iv*) *The equilibrium valuation of late stage start-up companies* V_2^* *is increasing in* ϕ_2*,* l_2 *and* k_2*, and decreasing in* σ_2*,* σ_2^V *and* δ_2*.*

The comparative statics results resemble those for the angel market equilibrium (see Proposition 1). We therefore do not provide a detailed discussion of the results here, and refer the reader to the explanations right after Proposition 1 in Section 3.1.

4 Cross-Market Effects

We now turn to the heart of our paper: analyzing the interrelationship between the angel and VC market. We first analyze the effects of angel market parameters on the VC market, and then look at the effects of VC parameters on the angel market.

From our characterization of the VC market equilibrium (see Appendix), we can see that the inflow of start-up companies, $\rho_1 x_1^*$, affects the absolute size of the VC market (M_2^{V*}), but not the equilibrium ratio of investors to companies (θ_2^*). In other words, when the inflow increases by x percent, then the number of VCs in the market also increases by x percent, so that the investor/company ratio remains constant in equilibrium. This is an important observation because, as we will show below, only the degree of competition in the VC market (θ_2^*) affects the angel market equilibrium, but not the absolute size of the VC market (M_2^{V*}). The equilibrium inflow of start-ups into the VC market, $\rho_1 x_1^*$, has therefore no feedback-loop effect back to the angel market. As shown in the Appendix, this property of our model also implies that the stationary equilibrium across the two markets is unique.

The next proposition provides a comprehensive characterization of how the equilibrium of the VC market depends on the parameters of the angel market.

Proposition 3 *Consider the VC market.*

(*i*) *The equilibrium market size* M_2^{V*} *is increasing in* ϕ_1 *and* σ_1^E*, and decreasing in* l_1*,* k_1*,* δ_1*, and* σ_1^A*.*

(*ii*) *The characteristics of the angel market do not affect the equilibrium degree of VC market competition* θ_2^**, the late stage deal rate* ξ_2^**, and the valuation of late stage start-ups,* V_2^**.*

We already know that a higher inflow of new ventures into the late stage market ($\rho_1 x_1^*$) leads to more VC entry, and therefore to a larger VC market. And the late stage inflow is determined by the number of deals in the early stage market, x_1^*. Noting that the early stage deal rate ξ_1^* is defined by $\xi_1^* = x_1^*/\overline{m}_1^E$, we can infer from Proposition 1 that the number of deals x_1^* is (i) increasing in the matching efficiency (ϕ_1) and in the entrepreneurs' search cost (σ_1^E), and (ii) decreasing in the death rate of early stage ventures (δ), the angels' search and investment costs (σ_1^A, k_1), and the entrepreneurs' labor input l_1.

Next we consider the effect of VC market parameters on the angel market. The equilibrium number of angels entering the early stage market in each period (m_1^{A*}), determines its absolute size and competitiveness (M_1^{A*}, θ_1^*). Moreover, the decision to enter the market depends on the expected utility from search in the late stage market, U_2^A. We show in the Appendix (see Proof of Proposition 2) that the equilibrium condition (6) – which defines U_2^A – can be written as

$$U_2^A = \frac{\sigma_2^V \theta_2^* - \sigma_2}{r + \delta_2}.$$

Thus, the angels' expected utility from search in the VC market, U_2^A, depends on the competitiveness of the VC market θ_2^*, but not on its absolute size M_2^{V*}.

We can now state how the angel market equilibrium depends on VC market parameters.

Proposition 4 *Consider the angel market. The size of the angel market M_1^{A*}, the equilibrium degree of competition θ_1^*, the early stage deal rate ξ_1^*, and the valuation of early stage start-ups V_1^*, are all*

(i) *increasing in ϕ_2, and*

(ii) *decreasing in δ_2, l_2, k_2, σ_2, and σ_2^V.*

Interestingly the effects on all of our four variables of interest – M_1^{A*}, θ_2^*, ξ_1^*, and V_1^* – go in the same direction. To see why, we note that entry of more angels has a positive effect on all of these variables. Moreover, a more competitive VC market means lower expected costs for securing the follow-on investment k_2, and therefore encourages more angel entry. We know from Proposition 2 that the VC market competition θ_2^* is increasing in ϕ_2, and decreasing in l_2, k_2, and σ_2^V. Furthermore, even though σ_2 and δ_2 both have a positive effect on the competitiveness of the VC market, we show in the Appendix that the net effect on U_2^A is negative (see Proof of Proposition 4). Accordingly, σ_2 and δ_2 have a negative effect on the equilibrium number of angels that enter the early stage market in each period.

It remains to be discussed briefly whether and how the late stage parameters affect the VC market equilibrium when the inflow of new ventures is endogenous, and therefore depends on the angel market characteristics. We know from Proposition 3 that the angel market characteristics only affect the absolute size of the VC market (M_2^{V*}), but not its competitiveness (θ_2^*), the deal rate (ξ_2^*), and the late stage valuations (V_2^*). For the latter three variables of interest (θ_2^*, ξ_2^*, V_2^*) the insights from Proposition 2 (within-market effects) therefore do not change. When considering each market separately, we know from Propositions 2 and 4 that the VC search cost σ_2^V, the required investment k_2, and the late stage labor input l_2, all have a negative effect on the number of investors in each market. When connecting both markets, we therefore find that these parameters reduce the inflow of new ventures into the late stage market, so that the VC market becomes smaller. Moreover, with separated markets, the late stage search cost for entrepreneurs and angels, σ_2, and the late stage death rate δ_2, both have opposite effects on the number of investors in the two markets. The net effect on the absolute size of the VC market M_2^{V*} is therefore ambiguous, and depends on the specific parameter values.

5 Hold-up of Angels

For the growth of their companies, angel investors rely on follow-on investments from VCs (k_2). This, and the fact that their early stage investments (k_1) are sunk, exposes angels to late stage hold-up by VCs. In this section we consider a simple model extension that allows us to further examine how any "hold-up" of angels affects the market equilibrium.[21]

We introduce a simple parameter that reflects the degree of angel hold-up in the late stage market; this allows us to derive comparative statics results for the equilibrium of the angel versus VC market. Suppose VCs and entrepreneurs can exclude angel investors in the late stage by incorporating a new venture based on the same business model. And suppose that the joint surplus for the subcoalition of the entrepreneur and VC (which excludes the angel investor) is given by $\lambda_A \pi$. The angel is indispensable when $\lambda_A = 0$; we considered this scenario in our base model. In contrast, the angel is completely dispensable when $\lambda_A = 1$. More generally λ_A measures how dispensable the angel is.

[21] The term "hold-up" should be interpreted with caution. From an ex-post perspective the angel investor (and the entrepreneur) may feel held-up, because they have made sunk investments, and because VCs can extract some bargaining surplus because of search frictions. However, from an ex-ante perspective, angels and entrepreneurs fully expect such hold-up. Moreover, there is free entry into both angel and VC markets, suggesting that all agents make fully informed participation decisions.

In the Appendix we derive the following new deal values (and related new equity shares) for the late stage market:

$$D_2^E = \frac{1}{6}[2+\lambda_A]\pi + \frac{1}{6}\left[U_2^E + U_2^A\right]$$

$$D_2^A = \frac{1}{3}[1-\lambda_A]\pi + \frac{1}{6}\left[U_2^E + U_2^A\right]$$

$$D_2^V = \frac{1}{6}[2+\lambda_A]\pi - \frac{1}{3}\left[U_2^E + U_2^A\right]$$

For now fix the expected joint utility from search for entrepreneurs and angels ($U_2^E + U_2^A$). We can then see that a higher degree of late stage hold-up (higher λ_A) reduces the angel's deal value (D_2^A), and improves the deal values for the VC and the entrepreneur (D_2^V and D_2^E). Being more dispensable weakens the angel's bargaining position, so that the VC and the entrepreneur both capture more of the surplus at the expense of the angel.[22] Of course, the equilibrium effect of the hold-up measure λ_A is more complex, as it also affects the outside options of entrepreneurs and angels (U_2^E, U_2^A).[23]

The next proposition sheds light on how late stage hold-up of angels affects the angel and VC market equilibrium.

Proposition 5 *The effect of late stage hold-up of angels, as measured by λ_A, is as follows:*

(i) ***Early stage:*** *The equilibrium size of the angel market M_1^{A*}, the degree of competition θ_1^*, and the early stage deal rate ξ_1^*, are all decreasing in the degree of late stage hold-up λ_A. The effect on the equilibrium valuation of early stage start-ups V_1^* is ambiguous.*

(ii) ***Late stage:*** *The equilibrium degree of competition θ_2^* and the late stage deal rate ξ_2^* are both increasing in the degree of late stage hold-up λ_A, while the valuation of late stage start-ups V_2^* is decreasing in λ_A. The effect on the equilibrium market size M_2^{V*} is ambiguous.*

Proposition shows that even though the hold-up is fully anticipated, and even though both markets have free entry, changes in the hold-up parameter λ_A affect the equilibrium of both markets. The anticipation of more severe late stage hold-up makes investments for angels less

[22]For tractability we continue to assume that the entrepreneur always needs the angel in case he has to search for a new VC. This leaves the angel with some bargaining power, so that his deal value D_2^A remains strictly positive even as $\lambda_A \to 1$.

[23]Introducing the hold-up parameter λ_A breaks the late stage symmetry between entrepreneurs and angels. This symmetry considerably simplified the derivations of our results for the base model. This is why we consider dispensable angels as a model extension.

attractive. As a consequence fewer angels enter the market, and the angel market becomes smaller and less competitive. A smaller number of early stage start-ups then receive angel financing. In equilibrium fewer ventures arrive in the VC market, thus limiting the deal flow for VCs. However, VCs can strike better deals as a result of hold-up, so investing in late stage start-ups becomes more attractive. This encourages more VC entry (relative to the deal inflow), and thus leads to a more competitive VC market. A larger number of late stage ventures then secures VC financing. Overall this provides two interesting implications: First, a more competitive late stage market limits the rents that VCs can extract when holding up angel investors. Second, while hold-up of angels is detrimental to early stage start-ups, new ventures that eventually make it to the VC market actually benefit from its more competitive nature: they can secure VC financing more quickly, and are thus less likely to become obsolete.

There is a negative effect of hold-up on the late stage valuation V_2^*. This is intuitive, since hold-up gives VCs better deal values, implying more equity (β^V) for themselves, and therefore lower valuations of late stage ventures. Concerning angel valuations, recall that the equilibrium equity share for an angel α^* – which determines the early stage valuation V_1^* – depends on the entrepreneur's outside option U_1^E. The effect of hold-up on U_1^E – and therefore on V_1^* – is twofold. First, a less competitive angel market implies that entrepreneurs search longer to find an angel investor, which has a negative impact on U_1^E. Second, entrepreneurs actually benefit from the hold-up of angels in the late stage market: directly through a higher deal value (D_2^E), and indirectly through a more competitive VC market (which reduces their expected search time). Which effect dominates depends on the market characteristics, in particular on the respective death rates in the early versus late stage market.

6 Early Exits

For our base model we assumed that the liquidation value of early stage ventures is sufficiently small, so that entrepreneurs and angels always want to seek VC financing. We now relax this assumption, and ask when entrepreneurs and angels avoid the VC market altogether, even though their ventures have growth options.

Consider again the early stage liquidation value y_1, and suppose that the support of this random variable is now $[0, y_2)$. Entrepreneurs and angels cannot liquidate their ventures unilaterally, and therefore need consent from their partners. In case they agree on an early exit, angels get $\alpha\widetilde{y}_1$ and entrepreneurs $(1-\alpha)\widetilde{y}_1$, where $\widetilde{y}_1$ denotes the realization of y_1. It is worth noting that the realized payoff $\widetilde{y}_1$ allows the partners to transfer utility, so that an early exit will always occur as long as it is Pareto improving. It is then easy to see that they always choose an early exit if

$$\widetilde{y}_1 \geq \widehat{y}_1 \equiv U_2^E + U_2^A.$$

The choice between an early exit and seeking VC financing depends on the joint expected utility from search in the late stage market. The next proposition sheds light on how VC market fundamentals affect the decision of entrepreneurs and angels to exit early.

Proposition 6 *The early exit threshold $\widehat{y}_1$ is increasing in ϕ_2, and decreasing in σ_2, δ_2, σ_2^V, l_2, and k_2.*

Entrepreneurs and angels avoid the VC market more often when securing follow-on investments is more costly for them (higher σ_2 and δ_2). The same applies when fewer VCs search for investment opportunities (due to higher σ_2^V and k_2), so that entrepreneurs and angels search on average longer before receiving VC. In equilibrium we observe more small entrepreneurial projects, and fewer ambitious projects that rely on large scale (VC) investments.

A related and interesting question is how hold-up of angels affects the decision to exit early. Consider again our model extension from Section 5, where the parameter λ_A measures how dispensable angels are in the late stage market. We obtain the following result:

Proposition 7 *More severe hold-up of angels in the late stage market leads to more early exits in equilibrium (i.e., $d\widehat{y}_1/d\lambda_A < 0$).*

We can immediately see from the modified late stage deal values in Section 5 that, ceteris paribus, more dispensable angels get a smaller share of the total surplus π, while entrepreneurs capture more of the surplus. We formally show in the Appendix (see Proof of Proposition 5) that this translates into a lower expected utility for angels (U_2^A), and a higher expected utility for entrepreneurs (U_2^E). However, VCs also capture a part of the surplus from angels, so that the net effect on their joint utility ($U_2^A + U_2^E$), and therefore $\widehat{y}_1$, is negative. Overall this implies that the anticipation of more severe hold-up of angels in the late stage market discourages owners of early stage ventures to exploit growth opportunities through raising VC. By some measure VCs shoot themselves in the foot: by holding up angel investors ex-post, they choke off their own deal flow ex-ante.

7 Capital- versus Labor-intense Projects

Entrepreneurial projects require two critical inputs: capital, provided by an outside investor (angel or VC), and labor, provided by the entrepreneur. In this section we ask how the intensity of capital versus labor affects the dynamics of the angel and VC markets.

Consider a set of projects that require the same total input Ψ_1 in the angel market, where $\Psi_1 \equiv k_1 + l_1$. The required labor input l_1 can then be expressed as a function of capital k_1:

$$l_1 = \Psi_1 - k_1. \tag{10}$$

We first identify the values of the required capital k_1 (relative to labor l_1) for which the early stage bargaining outcome is constrained or unconstrained. Using (10) we can write the equilibrium equity share for angels for the unconstrained case as

$$\alpha^* = \frac{1}{2} + \frac{2k_1 - \Psi_1 - U_1^E}{2(1-\rho_1)y_1}. \tag{11}$$

We note that the capital input k_1 has a direct and positive effect on the angel's equity share α^*. In the Appendix we show that U_1^E depends only on Ψ but not on k_1. Therefore α^* is strictly increasing in k_1, and we can define is inverse function, which we denote $\widehat{k}_1(\alpha^*)$, with $d\widehat{k}_1(\alpha^*)/d\alpha^* > 0$. We can distinguish between three regions. For $k_1 > \widehat{k}_1(1)$ we are in a highly capital-intensive region where $\alpha^* > 1$; for $\widehat{k}_1(0) < k_1 < \widehat{k}_1(1)$ we are in an intermediate region of capital labor intensity where $\alpha^* \in [0,1]$; and for $k_1 < \widehat{k}_1(0)$ we are in a highly labor-intensive region where $\alpha^* < 0$.

We define the unconstrained bargaining case as the case where the entrepreneur and angel implement the Nash bargaining outcome α^*. Unconstrained bargaining always obtains in the region with intermediate capital intensity, since it is always possible to set $\alpha = \alpha^* \in [0,1]$. However, as discussed below, the unconstrained bargaining may or may not apply for the highly capital- or highly labor-intensive regions.

For the unconstrained bargaining case we obtain the following proposition:

Proposition 8 *Under unconstrained bargaining the capital/labor intensity does not affect the equilibrium size of the angel market M_1^{A*}, the degree of competition θ_1^*, and the deal rate ξ_1^*. Start-ups with more capital intense projects have a higher post-money valuation V_1^* (i.e., $dV_1^*/dk_1 > 0$).*

The key intuition for this result is that the fundamental market parameters only depend on the overall profitability of the start-ups, but not on the specific mix of capital and labor inputs. In case of unconstrained Nash bargaining, the relative input contributions of the entrepreneur and angel are compensated through the allocation of equity shares. The more capital-intensive the project, the larger the share received by angel investors (higher α^*).

In the highly capital- or highly labor-intensive regions, the Nash bargaining outcome may be implementable, depending on whether the two parties have access to transferable utility or not. For $\alpha^* > 1$ the entrepreneur would need to transfer some utility to the angel by setting $\alpha = 1$ and then somehow transferring additional utility $(\alpha^* - 1)y_1$. Similarly, for $\alpha^* < 0$ the angel investor would set $\alpha = 0$ and then transfer additional utility $1 - \alpha^* y_1$ to the entrepreneur.

The most natural source of transferable utility would be prior wealth that can be used for transfer payments. In the case of $\alpha^* > 1$, the entrepreneur would use his wealth to co-invest alongside the angel investor. In the case of $\alpha^* < 0$, the angel would transfer wealth to the entrepreneur through a lump sum payment, which can be thought of as a salary. In practice such transfer payments are likely to be problematic. Entrepreneurs typically have insufficient wealth for co-investing; this is why they seek outside funding in the first place. As for angels, they may face severe adverse selection problems handing out lump sum payments to entrepreneurs.[24]

Another source of transferable utility might be the utility at the second stage. If the two parties can commit to a reward structure that differs from the late stage renegotiation outcome, then they can use this commitment to transfer utility. In the case of capital-intense projects, for example, the entrepreneur might want to commit to taking a smaller share in the late stage market, in order to make the deal more attractive to the angel investor. Again we note that this may be difficult in practice, as it remains unclear how entrepreneurs could credibly commit to this.

Based on this, it seems reasonable to entertain the possibility that the entrepreneur and investor do not have any (or sufficient) transferable utility. In this case it is impossible to implement the Nash bargaining solution α^* whenever $\alpha^* \notin [0, 1]$. The two parties then have to settle for a corner solution, namely $\alpha = 1$ ($\alpha = 0$) for capital (labor) intensive projects.

For a given set of parameters, the next proposition examines the equilibria under the constrained bargaining case, comparing them against the benchmark of the unconstrained bargaining case.

Proposition 9

(i) *For capital-intense projects (i.e., $k_1 > \widehat{k}_1(1)$), the equilibrium of the constrained bargaining case has an angel market that is smaller, less competitive, and with a lower deal rate, compared to the unconstrained case.*

(ii) *For labor-intense projects (i.e., $k_1 < \widehat{k}_1(0)$), the equilibrium of the constrained bargaining case has an angel market that is larger, more competitive, and with a higher deal rate, compared to the unconstrained case.*

The intuition for Proposition 9 is as follows. Compared to the unconstrained case, angels receive lower returns in the constrained model with capital-intense projects. This diminishes

[24] While not formally modeled here, it is easy to add a standard adverse selection problem to the model.

the deal values for angels, so that fewer angels enter the market in equilibrium. The opposite is true for labor-intense projects, where the constrained case has angels with higher deal values, which encourages more entry of angels.[25]

Proposition 9 does not address valuation. For $\alpha = 1$ the equilibrium valuation of early stage ventures reaches its lowest possible value at $V_1^* = k_1$. In this case the post-money valuation equals the investment amount, implying a pre-money valuation of zero. This may seems counter-intuitive at first. However, there exists an alternative more intuitive interpretation of this contract. Remember that α only represents the angel's stake in case of liquidation. In case of VC financing, the angel receives an equity stake commensurating with his bargaining power in the Shapley game. The $\alpha = 1$ contract resembles a security structure frequently used by angel investors, namely the convertible note. In case of liquidation the convertible note gives the angel a senior claim, typically amounting to the investment amount (or a multiple therefore), plus some accumulated dividends. This can be viewed as a senior claim on y_1. By setting a sufficiently high dividend rate or investment multiple, the investor gets a claim on the entire liquidation value, which is equivalent to $\alpha = 1$. In case of a VC round, however, the convertible note specifies a formula for how to calculate the angel's shares. In the model this formula can be set to match the expected outcome of the Shapley game. For the constrained bargaining case with capital intensive projects we can therefore interpret the optimal contract as a convertible note.

8 Conclusion

In this paper we develop a theory to explain the equilibrium interactions between angel and VC markets. We focus on the potential hold-up that angels may experience at the venture capital stage. Entry and competition in the venture capital market limit the extent of hold-up. Nonetheless we show that the hold-up reduces the returns to angel investments, and therefore reduces the flow of deals that emanate from the angel market.

Our analysis opens doors for further research into the relationship between early and late investors. One required assumption for the search model is the assumption of homogenous types. An interesting open issue is whether different types of investors and companies would be

[25]For completeness sake we also briefly discuss how the VC market equilibrium responds to capital- versus labor-intense projects. We first note that the total surplus π from the late stage investment can be written as $\pi = y_2 - \Psi_2$, with $\Psi_2 = k_2 + l_2$, which clearly does not depend on the specific capital/labor intensity. It is then easy to see that the equilibrium deal values D_2^i, $i = E, A, V$, do not depend on k_2 and l_2. This implies that the capital/labor intensity does not affect the equilibrium size of the VC market M_2^{V*}, the degree of competition θ_2^*, and the deal rate ξ_2^*. However, different levels of k_2 alter the equilibrium equity allocation, so that the valuation V_2^* is sensitive to the capital/labor intensity.

more or less exposed to hold-up. Another related interesting issue is to what extent investors can build reputations and networks that limit the extent of counter-productive hold-up. Finally, our framework raises an interesting policy question. If a government wanted to subsidize venture capital (presumably because of other market failures), would a subsidy to angels be more or less efficient than a subsidy to venture capitalists? We hope that future research, by ourselves and others, will help to illuminate these important derivative questions.

Appendix

Derivation of Angel Market Equilibrium.

Using $q_1^A = x_1/M_1^{A*}$ and $x_1 = \phi_1 \left[M_1^{E*} M_1^{A*}\right]^{0.5}$, we can write (3) as

$$\frac{\phi_1 (M_1^{E*})^{0.5}}{(M_1^{A*})^{0.5}} \widetilde{D}_1^A = \sigma_1^A. \tag{12}$$

Using $\theta_1^* = M_1^{A*}/M_1^{E*}$, we get the equilibrium degree of competition for the angel market: $\theta_1^* = \left[\phi_1 \widetilde{D}_1^A / \sigma_1^A\right]^2$. Next, note that we can write (12) as

$$M_1^{A*} = M_1^{E*} \left[\frac{\phi_1 \widetilde{D}_1^A}{\sigma_1^A}\right]^2 = M_1^{E*} \theta_1^*.$$

Solving (4) for M_1^{E*}, and using $q_1^E = \phi_1 \left[M_1^{E*} M_1^{A*}\right]^{0.5} / M_1^{E*} = \phi_1 \left[M_1^{A*}/M_1^{E*}\right]^{0.5}$, we get

$$M_1^{E*} = \frac{\overline{m}_1^E}{\delta_1 + q_1^E} = \frac{\overline{m}_1^E}{\delta_1 + \phi_1 \left[M_1^{A*}/M_1^{E*}\right]^{0.5}} = \frac{\overline{m}_1^E}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}.$$

Thus, the equilibrium size of the angel market is

$$M_1^{A*} = M_1^{E*} \theta_1^* = \frac{\overline{m}_1^E \theta_1^*}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}.$$

Next, recall that the equilibrium early stage deal rate is given by $\xi_1^* = x_1^*/\overline{m}_1^E$. Using $M_1^{E*} = M_1^{A*}/\theta_1^*$ we can write x_1^* as

$$x_1^* = \phi_1 \left[M_1^{A*} M_1^{E*}\right]^{0.5} = \frac{\phi_1 M_1^{A*}}{[\theta_1^*]^{0.5}} = \frac{\overline{m}_1^E \phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}.$$

Thus,

$$\xi_1^* = \frac{\phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}.$$

Finally, recall that the equilibrium valuation V_1^* is defined by $V_1^* = k_1/\alpha^*$. Using (1) we get

$$V_1^* = \frac{2k_1(1-\rho_1)y_1}{(1-\rho_1)y_1 + k_1 - l_1 - U_1^E}.$$

Proof of Proposition 1.

Consider the equilibrium degree of competition θ_1^*. We first note that

$$\frac{d\theta_1^*}{dU_1^E} = 2\frac{\phi_1 \widetilde{D}_1^A}{\sigma_1^A}\frac{d\widetilde{D}_1^A}{dU_1^E} = -\frac{\phi_1 \widetilde{D}_1^A}{\sigma_1^A} < 0.$$

Next, we derive a condition which defines U_1^E. The equilibrium condition (2) can be written as

$$U_1^E\left[r+\delta_1\right] = -\sigma_1^E + q_1^E\left[\widetilde{D}_1^E - U_1^E\right].$$

It is easy to see that $\widetilde{D}_1^E - U_1^E = \widetilde{D}_1^A$. Using $q_1^E = \phi_1\left[M_1^{A*}/M_1^{E*}\right]^{0.5} = \phi_1\sqrt{\theta_1^*} = \phi_1^2\widetilde{D}_1^A/\sigma_1^A$ we get the following condition which defines U_1^E:

$$U_1^E\left[r+\delta_1\right] - \frac{\phi_1^2}{\sigma_1^A}\left[\widetilde{D}_1^A\right]^2 + \sigma_1^E = 0. \tag{13}$$

Note that δ_1 only affects U_1^E in the definition of θ_1^*. Implicitly differentiating U_1^E w.r.t. δ_1 yields

$$\frac{dU_1^E}{d\delta_1} = -\frac{U_1^E}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} < 0.$$

Thus, $d\theta_1^*/d\delta_1 > 0$. Likewise, σ_1^E only affects U_1^E in the definition of θ_1^*. We get

$$\frac{dU_1^E}{d\sigma_1^E} = -\frac{1}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} < 0,$$

which implies $d\theta_1^*/d\sigma_1^E > 0$. Next, implicitly differentiating θ_1^* w.r.t. ϕ_1 yields

$$\frac{d\theta_1^*}{d\phi_1} = 2\left[\frac{\phi_1\widetilde{D}_1^A}{\sigma_1^A}\right]\left[\frac{\widetilde{D}_1^A}{\sigma_1^A} - \frac{\phi_1}{2\sigma_1^A}\frac{dU_1^E}{d\phi_1}\right], \text{ with } \frac{dU_1^E}{d\phi_1} = \frac{2\frac{\phi_1}{\sigma_1^A}\left[\widetilde{D}_1^A\right]^2}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} > 0.$$

Thus,

$$\frac{d\theta_1^*}{d\phi_1} = 2\frac{\phi_1\widetilde{D}_1^A}{\sigma_1^A}\frac{2\widetilde{D}_1^A(r+\delta_1)}{2\sigma_1^A\left[r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A\right]} > 0.$$

Likewise, differentiating θ_1^* w.r.t. σ_1^A we find

$$\frac{d\theta_1^*}{d\sigma_1^A} = 2\left[\frac{\phi_1 \widetilde{D}_1^A}{\sigma_1^A}\right]\left[-\frac{\phi_1 \widetilde{D}_1^A}{\left[\sigma_1^A\right]^2} - \frac{\phi_1}{2\sigma_1^A}\frac{dU_1^E}{d\sigma_1^A}\right], \text{ with } \frac{dU_1^E}{d\sigma_1^A} = \frac{\left[\frac{\phi_1}{\sigma_1^A}\widetilde{D}_1^A\right]^2}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} > 0.$$

Consequently,

$$\frac{d\theta_1^*}{d\sigma_1^A} = 2\left[\frac{\phi_1 \widetilde{D}_1^A}{\sigma_1^A}\right]\left[\frac{-2\phi_1 \widetilde{D}_1^A\left[r + \delta_1\right] - \frac{\phi_1^3}{\sigma_1^A}\left[\widetilde{D}_1^A\right]^2}{2\left[\sigma_1^A\right]^2\left[r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A\right]}\right] < 0.$$

Next we differentiate θ_1^* w.r.t. l_1:

$$\frac{d\theta_1^*}{dl_1} = -\frac{\phi_1}{\sigma_1^A}\widetilde{D}_1^A\left[1 + \frac{dU_1^E}{dl_1}\right], \text{ with } \frac{dU_1^E}{dl_1} = -\frac{\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}.$$

Thus,

$$\frac{d\theta_1^*}{dl_1} = -\frac{\phi_1}{\sigma_1^A}\widetilde{D}_1^A\left[1 - \frac{\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}\right] = -\frac{\phi_1}{\sigma_1^A}\widetilde{D}_1^A\left[\frac{r + \delta_1}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}\right] < 0.$$

Finally, differentiating θ_1^* w.r.t. k_1 yields

$$\frac{d\theta_1^*}{dk_1} = 2\left[\frac{\phi_1 \widetilde{D}_1^A}{\sigma_1^A}\right]\frac{\phi_1}{\sigma_1^A}\left[-\frac{1}{2} - \frac{1}{2}\frac{dU_1^E}{dk_1}\right], \text{ with } \frac{dU_1^E}{dk_1} = -\frac{\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} < 0.$$

Thus,

$$\frac{d\theta_1^*}{dk_1} = \left[\frac{\phi_1 \widetilde{D}_1^A}{\sigma_1^A}\right]\frac{\phi_1}{\sigma_1^A}\left[\frac{-(r + \delta_1)}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}\right] < 0.$$

Now consider the equilibrium size of the angel market M_1^{A*}. Note that

$$\frac{dM_1^{A*}}{d\theta_1^*} = \frac{\overline{m}_1^E\left[\delta_1 + \phi_1\left(\theta_1^*\right)^{0.5}\right] - \frac{1}{2}\overline{m}_1^E\theta_1^*\phi_1\left(\theta_1^*\right)^{-0.5}}{\left[\delta_1 + \phi_1\left(\theta_1^*\right)^{0.5}\right]^2} = \frac{\overline{m}_1^E\left[\delta_1 + \frac{1}{2}\phi_1\left(\theta_1^*\right)^{0.5}\right]}{\left[\delta_1 + \phi_1\left(\theta_1^*\right)^{0.5}\right]^2} > 0.$$

Our comparative statics results for θ_1^* then imply that $dM_1^{A*}/d\sigma_1^E > 0$, and $dM_1^{A*}/d\sigma_1^A$, dM_1^{V*}/dl_1, $dM_1^{A*}/dk_1 < 0$. However, it is easy to verify that the signs of $dM_1^{A*}/d\phi_1$ and $dM_1^{A*}/d\delta_1$ are ambiguous, and depend on the specific parameter values.

Now consider the equilibrium deal rate ξ_1^*. We first note that

$$\frac{d\xi_1^*}{d(\phi_1\sqrt{\theta_1^*})} = \rho_1 \frac{\delta_1}{\left[\delta_1 + \phi_1\sqrt{\theta_1^*}\right]^2} > 0.$$

Our comparative statics results for θ_1^* then immediately imply that $d\xi_1^*/d\phi_1$, $d\xi_1^*/d\sigma_1^E > 0$, and $d\xi_1^*/dl_1$, $d\xi_1^*/dk_1$, $d\xi_1^*/d\sigma_1^A < 0$. To show that $d\xi_1^*/d\delta_1 < 0$, it is convenient to rewrite ξ_1^* as follows:

$$\xi_1^* = \frac{1}{1 + \delta_1\left[\phi_1\sqrt{\theta_1^*}\right]^{-1}}.$$

Thus, $d\xi_1^*/d\delta_1 < 0$ if

$$\frac{d}{d\delta_1}\left(\frac{\delta_1}{\phi_1\sqrt{\theta_1^*}}\right) = \frac{d}{d\delta_1}\left(\frac{\delta_1}{\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}\right) > 0.$$

Let $a_1 \equiv \phi_1^2/\sigma_1^A$. Using the definition of $\widetilde{D}_1^A$ we get

$$\frac{d}{d\delta_1}\left(\frac{\delta_1}{a_1\widetilde{D}_1^A}\right) = \frac{a_1\widetilde{D}_1^A + \frac{1}{2}\delta_1 a_1 \frac{dU_1^E}{d\delta_1}}{\left[a_1\widetilde{D}_1^A\right]^2} = \frac{a_1\widetilde{D}_1^A - \frac{1}{2}\delta_1 a_1 \frac{U_1^E}{r+\delta_1+a_1\widetilde{D}_1^A}}{\left[a_1\widetilde{D}_1^A\right]^2},$$

which is strictly positive if

$$a_1\widetilde{D}_1^A \;>\; \frac{1}{2}\delta_1 a_1 \frac{U_1^E}{r+\delta_1+a_1\widetilde{D}_1^A} \tag{14}$$

$$\Leftrightarrow \quad 2\widetilde{D}_1^A\left[r+\delta_1+a_1\widetilde{D}_1^A\right] \;>\; U_1^E\left[r+\delta_1\right] - rU_1^E \tag{15}$$

Using (13) we can write condition (15) as

$$2\widetilde{D}_1^A\left[r+\delta_1+a_1\widetilde{D}_1^A\right] \;>\; a_1\left[\widetilde{D}_1^A\right]^2 - \sigma_1^E - rU_1^E$$

$$\Leftrightarrow \quad 2\widetilde{D}_1^A\left[r+\delta_1\right] + a_1\left[\widetilde{D}_1^A\right]^2 \;>\; -\sigma_1^E - rU_1^E,$$

which is clearly satisfied. Thus, $d\xi_1^*/d\delta_1 < 0$.

Finally consider the equilibrium valuation V_1^*. Note that $dV_1^*/dU_1^E > 0$; this immediately implies that $dV_1^*/d\phi_1 > 0$ and $dV_1^*/d\sigma_1^E$, $dV_1^*/d\sigma_1^A$, $dV_1^*/d\delta_1 < 0$. Moreover,

$$\frac{dV_1^*}{dl_1} = \frac{2k_1(1-\rho_1)y_1\left[1+\frac{dU_1^E}{dl_1}\right]}{\left[(1-\rho_1)y_1+k_1-l_1-U_1^E\right]^2} = \frac{2k_1(1-\rho_1)y_1\left[1+\frac{r+\delta_1}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}\right]}{\left[(1-\rho_1)y_1+k_1-l_1-U_1^E\right]^2} > 0.$$

Recall that $V_1^* = k_1/\alpha^*$. Taking the first derivative w.r.t. k_1 we get

$$\frac{dV_1^*}{dk_1} = \frac{\overbrace{\alpha^* - k_1\frac{d\alpha^*}{dk_1}}^{\equiv N}}{\left[\alpha^*\right]^2}.$$

The denominator is always non-negative. Moreover, note that $N \geq 0$ for $k_1 \to 0$, which implies that $dV_1^*/dk_1 \geq 0$ for $k_1 \to 0$. To show that $dV_1^*/dk_1 > 0$ for all $k_1 > 0$, it is thus sufficient to verify that $dN/dk_1 > 0$:

$$\frac{dN}{dk_1} = \frac{d\alpha^*}{dk_1} - \left(\frac{d\alpha^*}{dk_1} + k_1\frac{d^2\alpha^*}{dk_1^2}\right) = -k_1\frac{d^2\alpha^*}{dk_1^2}.$$

Next, we need to find the sign of $d^2\alpha^*/dk_1^*$. We start by taking the first derivative of α^* w.r.t. k_1:

$$\frac{d\alpha^*}{dk_1} = \frac{1}{2(1-\rho_1)y_1}\left[1-\frac{dU_1^E}{dk_1}\right] = \frac{1}{2(1-\rho_1)y_1}\left[1+\frac{1}{(r+\delta_1)\left[a_1\widetilde{D}_1^A\right]^{-1}+1}\right],$$

where $a_1 = \phi_1^2/\sigma_1^A$. Consequently,

$$\frac{d^2\alpha^*}{dk_1^2} = \frac{1}{2(1-\rho_1)y_1}\frac{-\frac{1}{2}a_1\left(r+\delta_1\right)\left[a_1\widetilde{D}_1^A\right]^{-2}\left[1+\frac{dU_1^E}{dk_1}\right]}{\left[\left(r+\delta_1\right)\left[a_1\widetilde{D}_1^A\right]^{-1}+1\right]^2},$$

where

$$1+\frac{dU_1^E}{dk_1} = 1-\frac{a_1\widetilde{D}_1^A}{r+\delta_1+a_1\widetilde{D}_1^A} = \frac{r+\delta_1}{r+\delta_1+a_1\widetilde{D}_1^A} > 0.$$

Thus, $d^2\alpha^*/dk_1^2 < 0$. This implies that $dN/dk_1 > 0$, and therefore $dV_1^*/dk_1 > 0$. □

Derivation of Deal Values and Equity Shares – VC Market.

Let CV_i denote the value generated by the coalition $i = EAV, EV, EA, AV, E, A, V$. Using the Shapley value, we get the following general deal values from the tripartite bargaining game:

$$D_2^E = \frac{1}{3}[CV_{EAV} - CV_{AV}] + \frac{1}{6}[CV_{EA} - CV_A] + \frac{1}{6}[CV_{EV} - CV_V] + \frac{1}{3}CV_E \quad (16)$$

$$D_2^A = \frac{1}{3}[CV_{EAV} - CV_{EV}] + \frac{1}{6}[CV_{EA} - CV_E] + \frac{1}{6}[CV_{AV} - CV_V] + \frac{1}{3}CV_A \quad (17)$$

$$D_2^V = \frac{1}{3}[CV_{EAV} - CV_{EA}] + \frac{1}{6}[CV_{EV} - CV_E] + \frac{1}{6}[CV_{AV} - CV_A] + \frac{1}{3}CV_V \quad (18)$$

We note that $CV_{EAV} = \pi$, $CV_{EA} = U_2^E + U_2^A$, and $CV_{AV} = CV_{EV} = CV_E = CV_A = CV_V = 0$. Thus,

$$D_2^E = \frac{1}{3}\pi + \frac{1}{6}\left[U_2^E + U_2^A\right]$$

$$D_2^A = \frac{1}{3}\pi + \frac{1}{6}\left[U_2^E + U_2^A\right]$$

$$D_2^V = \frac{1}{3}\pi - \frac{1}{3}\left[U_2^E + U_2^A\right]$$

The deal values then allow us to derive the equilibrium equity shares β^{E*}, β^{A*}, and β^{V*}. The equilibrium equity share for entrepreneurs, β^{E*}, ensures that their actual net payoff equals their deal value from the bargaining game:[26]

$$\beta^{E*} y_2 - l_2 = D_2^E. \quad (19)$$

Solving (19) for β^{E*} yields

$$\beta^{E*} = \frac{l_2 + D_2^E}{y_2} = \frac{1}{6y_2}\left[6l_2 + 2\pi + U_2^E + U_2^A\right].$$

[26] It is important here to note that late stage ventures will never be liquidated in equilibrium, so the payoff from a venture is always y_2 once a VC invests k_2.

Likewise we get

$$\beta^{A*} = \frac{D_2^A}{y_2} = \frac{1}{6y_2}\left[2\pi + U_2^E + U_2^A\right]$$
$$\beta^{V*} = \frac{k_2 + D_2^V}{y_2} = \frac{1}{3y_2}\left[3k_2 + \pi - \left(U_2^E + U_2^A\right)\right].$$

Derivation of VC Market Equilibrium.

The first part of the derivation follows along the lines of the derivation of the angel market equilibrium: Using (7) we get $\theta_2^* = \left[\phi_2 D_2^V/\sigma_2^V\right]^2$. Moreover, using the relationship $M_2^V = M_2^E\theta_2^*$ we find

$$M_2^{V*} = \frac{\rho_1 x_1^* \theta_2^*}{\delta_2 + \phi_2\sqrt{\theta_2^*}}.$$

Recall that the equilibrium late stage deal rate is given by $\xi_2^* = x_2^*/\left(\rho_1 x_1^*\right)$. Using $M_2^{E*} = M_2^{V*}/\theta_2^*$ and the definition of M_2^{V*}, we can write x_2^* as

$$x_2^* = \phi_2\left[M_2^{V*}M_2^{E*}\right]^{0.5} = \frac{\phi_2 M_2^{V*}}{\left[\theta_2^*\right]^{0.5}} = \frac{\rho_1 x_1^* \phi_2\sqrt{\theta_2^*}}{\delta_2 + \phi_2\sqrt{\theta_2^*}}.$$

Thus,

$$\xi_2^* = \frac{\phi_2\sqrt{\theta_2^*}}{\delta_2 + \phi_2\sqrt{\theta_2^*}}.$$

Finally, using the equilibrium equity share β^{V*} for VCs, we can write V_2^* as follows:

$$V_2^* = \frac{k_2}{\beta^{V*}} = \frac{k_2 y_2}{k_2 + D_2^V} = \left(\frac{3k_2}{3k_2 + \pi - (U_2^E + U_2^A)}\right) y_2.$$

Proof of Proposition 2.

Consider the equilibrium degree of competition θ_2^*. Recall that $U_2^A = U_2^E$ in equilibrium; thus,

$$\frac{d\theta_2^*}{dU_2^E} = \frac{d\theta_2^*}{dU_2^A} = 2\frac{\phi_2 D_2^V}{\sigma_2^V}\frac{dD_2^V}{dU_2^E} = -\frac{4}{3}\frac{\phi_2 D_2^V}{\sigma_2^V} < 0.$$

Next, we derive a condition which defines U_2^E. We can write (6) as

$$U_2^E\left[r + \delta_2\right] = -\sigma_2 + q_2^E\left[D_2^E - U_2^E\right].$$

Note that $D_2^E - U_2^E = \pi/3 - 2U_2^E/3 = D_2^V$. Using $q_2^E = \phi_2 \left[M_2^{V*}/M_2^{E*}\right]^{0.5} = \phi_2 \sqrt{\theta_2^*} = \phi_2^2 D_2^V / \sigma_2^V$, we get the following condition which defines U_2^E:

$$U_2^E \left[r + \delta_2\right] - \frac{\phi_2^2}{\sigma_2^V} \left[D_2^V\right]^2 + \sigma_2 = 0. \tag{20}$$

Note that (20) has the same structure as (13), which defines U_1^E. Thus, following along the lines of Proof of Proposition 1 it is straightforward to show that $d\theta_2^*/d\delta_2$, $d\theta_2^*/d\sigma_2$, $d\theta_2^*/d\phi_2 > 0$, and $d\theta_2^*/d\sigma_2^V$, $d\theta_2^*/dl_2$, $d\theta_2^*/dk_2 < 0$.

Now consider the equilibrium size of the VC market M_2^{V*}. Note that

$$\frac{dM_2^{V*}}{d\theta_2^*} = \frac{\rho_1 x_1^* \left[\delta_2 + \phi_2 \left(\theta_2^*\right)^{0.5}\right] - \frac{1}{2}\rho_1 x_1^* \theta_2^* \phi_2 \left(\theta_2^*\right)^{-0.5}}{\left[\delta_2 + \phi_2 \left(\theta_2^*\right)^{0.5}\right]^2} = \frac{\rho_1 x_1^* \left[\delta_2 + \frac{1}{2}\phi_2 \left(\theta_2^*\right)^{0.5}\right]}{\left[\delta_2 + \phi_2 \left(\theta_2^*\right)^{0.5}\right]^2} > 0.$$

Our comparative statics results for θ_2^* then imply that $dM_2^{V*}/d\sigma_2 > 0$, and $dM_2^{V*}/d\sigma_2^V$, dM_2^{V*}/dl_2, $dM_2^{V*}/dk_2 < 0$. Moreover, it is straightforward to verify that the signs of $dM_2^{V*}/d\phi_2$ and $dM_2^{V*}/d\delta_2$ are ambiguous, and depend on the specific parameter values.

Now consider the equilibrium late stage deal rate ξ_2^*. Note that

$$\frac{d\xi_2^*}{d(\phi_2\sqrt{\theta_2^*})} = \frac{\delta_2}{\left[\delta_2 + \phi_2\sqrt{\theta_2^*}\right]^2} > 0.$$

Our comparative statics results for θ_2^* then imply that $d\xi_2^*/d\phi_2$, $d\xi_2^*/d\sigma_2 > 0$, and $d\xi_2^*/dl_2$, $d\xi_2^*/dk_2$, $d\xi_2^*/d\sigma_2^V < 0$. To prove that $d\xi_2^*/d\delta_2 < 0$, we rewrite ξ_2^* as follows:

$$\xi_2^* = \frac{1}{1 + \delta_2 \left[\phi_2\sqrt{\theta_2^*}\right]^{-1}}.$$

Thus, $d\xi_2^*/d\delta_2 < 0$ if

$$\frac{d}{d\delta_2}\left(\frac{\delta_2}{\phi_2\sqrt{\theta_2^*}}\right) = \frac{d}{d\delta_2}\left(\frac{\delta_2}{\frac{\phi_2^2}{\sigma_2^V} D_2^V}\right) > 0.$$

For parsimony we define $a_2 \equiv \phi_2^2/\sigma_2^V$. We get

$$\frac{d}{d\delta_2}\left(\frac{\delta_2}{a_2 D_2^V}\right) = \frac{a_2 D_2^V + \frac{2}{3}\delta_2 a_2 \frac{dU_2^E}{d\delta_2}}{\left[a_2 D_2^V\right]^2} = \frac{a_2 D_2^V - \frac{2}{3}\delta_2 a_2 \frac{U_2^E}{r+\delta_2+\frac{4}{3}a_2 D_2^V}}{\left[a_2 D_2^V\right]^2},$$

which is strictly positive if

$$
\begin{aligned}
a_2 D_2^V &> \frac{2}{3}\delta_2 a_2 \frac{U_2^E}{r+\delta_2+\frac{4}{3}a_2 D_2^V} \\
\Leftrightarrow \quad \frac{3}{2}D_2^V\left[r+\delta_2+\frac{4}{3}a_2 D_2^V\right] &> U_2^E\left[r+\delta_2\right]-rU_2^E
\end{aligned}
$$

Using (20) we can write this condition as

$$
\begin{aligned}
\frac{3}{2}D_2^V\left[r+\delta_2+\frac{4}{3}a_2 D_2^V\right] &> a_2\left[D_2^V\right]^2-\sigma_2-rU_2^E \\
\Leftrightarrow \quad \frac{3}{2}D_2^V\left[r+\delta_2\right]+a_2\left[D_2^V\right]^2 &> -\sigma_2-rU_2^E,
\end{aligned}
$$

which is clearly satisfied. Thus, $d\xi_2^*/d\delta_2 < 0$.

Finally consider the equilibrium valuation $V_2^* = k_2 y_2/(k_2+D_2^V)$. Note that $dV_2^*/dD_2^V < 0$. Moreover, $dD_2^V/dU_2^E < 0$, which implies that $dV_2^*/dU_2^E > 0$. Following along the lines of Proof of Proposition 1 it is straightforward to show that $dU_2^E/d\phi_2 > 0$, and $dU_2^E/d\sigma_2$, $dU_2^E/d\sigma_2^V$, $dU_2^E/d\delta_2 < 0$. Consequently, $dV_2^*/d\phi_2 > 0$ and $dV_2^*/d\sigma_2$, $dV_2^*/d\sigma_2^V$, $dV_2^*/d\delta_2 < 0$. Furthermore,

$$
\frac{dD_2^V}{dl_2} = -\frac{1}{3}-\frac{2}{3}\frac{dU_2^E}{dl_2}, \text{ with } \frac{dU_2^E}{dl_2} = -\frac{\frac{2}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V}.
$$

Thus,

$$
\frac{dD_2^V}{dl_2} = \frac{1}{3}\left[-1+\frac{\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V}\right] = -\frac{1}{3}\left[\frac{r+\delta_2}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V}\right] < 0.
$$

This implies that $dV_2^*/dl_2 > 0$. Next, taking the first derivative of V_2^* w.r.t. k_2 yields

$$
\frac{dV_2^*}{dk_2} = \frac{k_2+D_2^V-k_2\left[1-\frac{1}{3}-\frac{2}{3}\frac{dU_2^A}{dk_2}\right]}{\left[k_2+D_2^V\right]^2}y_2 = \frac{\overbrace{\frac{1}{3}k_2+D_2^V+\frac{2}{3}k_2\frac{dU_2^A}{dk_2}}^{\equiv N}}{\left[k_2+D_2^V\right]^2}y_2.
$$

Note that the denominator is always positive. Moreover, we have $N > 0$ for $k_2 \to 0$. Thus, $dV_2^*/dk_2 > 0$ for $k_2 \to 0$. To verify that $dV_2^*/dk_2 > 0$ for all $k_2 > 0$, it is therefore sufficient to show that $dN/dk_2 > 0$:

$$\frac{dN}{dk_2} = \frac{1}{3} - \frac{1}{3} - \frac{2}{3}\frac{dU_2^A}{dk_2} + \frac{2}{3}\left[\frac{dU_2^A}{dk_2} + k_2\frac{d^2U_2^A}{dk_2^2}\right] = \frac{2}{3}k_2\frac{d^2U_2^A}{dk_2^2}.$$

It remains to identify the sign of $d^2U_2^A/dk_2^2$. We first take the derivative of U_2^A w.r.t. k_2:

$$\frac{dU_2^A}{dk_2} = -\frac{\frac{2}{3}a_2D_2^V}{r+\delta_2+\frac{4}{3}a_2D_2^V} = -\frac{\frac{2}{3}}{(r+\delta_2)\left[a_2D_2^V\right]^{-1}+\frac{4}{3}},$$

where $a_2 = \phi_2^2/\sigma_2^V$. Thus,

$$\frac{d^2U_2^A}{dk_2^2} = \frac{\frac{2}{9}a_2\left(r+\delta_2\right)\left[a_2D_2^V\right]^{-2}\left[1+2\frac{dU_2^A}{dk_2}\right]}{\left[(r+\delta_2)\left[a_2D_2^V\right]^{-1}+\frac{4}{3}\right]^2}.$$

Note that

$$1+2\frac{dU_2^A}{dk_2} = 1-\frac{\frac{4}{3}a_2D_2^V}{r+\delta_2+\frac{4}{3}a_2D_2^V} = \frac{r+\delta_2}{r+\delta_2+\frac{4}{3}a_2D_2^V} > 0.$$

Hence, $d^2U_2^A/dk_2^2 > 0$, so that $dN/dk_2 > 0$. Consequently, $dV_2^*/dk_2 > 0$. □

Proof of Proposition 3.

We can see from (20) that U_2^E does not depend on the early stage parameters ϕ_1, σ_1^A, σ_1^E, l_1, k_1, δ_1. This also implies that D_2^V – and therefore θ_2^*, V_2^*, and ξ_2^* – do not depend on these parameters. Next, we note that $dM_2^{V*}/dx_1^* > 0$. Moreover, using the definition of x_1^* (see the derivation of the angel market equilibrium) we get

$$\frac{dx_1^*}{d\left(\phi_1\sqrt{\theta_1^*}\right)} = \frac{\overline{m}_1^E\left[\delta_1+\phi_1\sqrt{\theta_1^*}\right]-\overline{m}_1^E\phi_1\sqrt{\theta_1^*}}{\left[\delta_1+\phi_1\sqrt{\theta_1^*}\right]^2} = \frac{\overline{m}_1^E\delta_1}{\left[\delta_1+\phi_1\sqrt{\theta_1^*}\right]^2} > 0.$$

Because $d\theta_1^*/d\phi_1 > 0$ we have $dx_1^*/d\phi_1 > 0$, and therefore, $dM_2^{V*}/d\phi_1 > 0$. Moreover, our comparative statics results for θ_1^* (see Proposition 1) imply that $dM_2^{V*}/d\sigma_1^E > 0$, and dM_2^{V*}/dl_1,

dM_2^{V*}/dk_1, $dM_2^{V*}/d\sigma_1^A < 0$. Next we show that $dx_1^*/d\delta_1 < 0$, so that $dM_2^{V*}/d\delta_1 < 0$. Using $\theta_1^* = \left[\phi_1 \widetilde{D}_1^A/\sigma_1^A\right]^2$ we can write x_1^* as

$$x_1^* = \frac{\overline{m}_1^E a_1 \widetilde{D}_1^A}{\delta_1 + a_1 \widetilde{D}_1^A},$$

where $a_1 = \phi_1^2/\sigma_1^A$. The first derivative of x_1^* w.r.t. δ_1 is

$$\frac{dx_1^*}{d\delta_1} = \frac{\overline{m}_1^E a_1 \frac{d\widetilde{D}_1^A}{d\delta_1}\left[\delta_1 + a_1 \widetilde{D}_1^A\right] - \overline{m}_1^E a_1 \widetilde{D}_1^A \left[1 + a_1 \frac{d\widetilde{D}_1^A}{d\delta_1}\right]}{\left[\delta_1 + a_1 \widetilde{D}_1^A\right]^2}.$$

Note that $d\widetilde{D}_1^A/d\delta_1 = -(dU_1^E/d\delta_1)/2$. Thus, $dx_1^*/d\delta_1 < 0$ if

$$\frac{d\widetilde{D}_1^A}{d\delta_1}\left[\delta_1 + a_1 \widetilde{D}_1^A\right] < \widetilde{D}_1^A \left[1 + a_1 \frac{d\widetilde{D}_1^A}{d\delta_1}\right] \quad \Leftrightarrow \quad -\frac{1}{2}\frac{dU_1^E}{d\delta_1}\delta_1 < \widetilde{D}_1^A.$$

Recall from Proof of Proposition 1 that

$$\frac{dU_1^E}{d\delta_1} = -\frac{U_1^E}{r + \delta_1 + a_1 \widetilde{D}_1^A}. \tag{21}$$

Using this we can write the condition for $dx_1^*/d\delta_1 < 0$ as follows:

$$\underbrace{2\widetilde{D}_1^A \left[r + \delta_1 + a_1 \widetilde{D}_1^A\right] - U_1^E \delta_1}_{\equiv N} > 0. \tag{22}$$

Note that this condition is satisfied for $\delta_1 = 0$. To show that (22) holds for all $\delta_1 > 0$, it is therefore sufficient to verify that N is increasing in δ_1:

$$\frac{dN}{d\delta_1} = -\frac{dU_1^E}{d\delta_1}\left[r + \delta_1 + a_1 \widetilde{D}_1^A\right] - \widetilde{D}_1^A a_1 \frac{dU_1^E}{d\delta_1} - \delta_1 \frac{dU_1^E}{d\delta_1} - U_1^E.$$

Using (21) we find

$$U_1^E = -\frac{dU_1^E}{d\delta_1}\left[r + \delta_1 + a_1 \widetilde{D}_1^A\right].$$

We can then use this relationship to simplify $dN/d\delta_1$:

$$\frac{dN}{d\delta_1} = -\widetilde{D}_1^A a_1 \frac{dU_1^E}{d\delta_1} - \delta_1 \frac{dU_1^E}{d\delta_1},$$

which is strictly positive since $dU_1^E/d\delta_1 < 0$. This implies that $dx_1^*/d\delta_1 < 0$, and therefore $dM_2^{V*}/d\delta_1 < 0$. $\square$

Proof of Proposition 4.

In equilibrium, $U_2^E = U_2^A$. Thus, we can write $\widetilde{D}_1^A$ as

$$\widetilde{D}_1^A = \frac{1}{2}\left[2\rho_1 U_2^A + (1-\rho_1)y_1 - k_1 - l_1 - U_1^E\right].$$

Differentiating $\widetilde{D}_1^A$ w.r.t. U_2^A yields

$$\frac{d\widetilde{D}_1^A}{dU_2^A} = \rho_1 - \frac{1}{2}\frac{dU_1^E}{dU_2^A}, \text{ with } \frac{dU_1^E}{dU_2^A} = \frac{2\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A\rho_1}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} > 0.$$

Thus,

$$\frac{d\widetilde{D}_1^A}{dU_2^A} = \rho_1 - \frac{\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A\rho_1}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} = \frac{\rho_1\left[r+\delta_1\right]}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} > 0.$$

This implies that $d\theta_1^*/dU_2^A > 0$ and $dM_1^{A*}/dU_2^A > 0$ (as $dM_1^{A*}/d\theta_1^* > 0$; see Proof of Proposition 1). Moreover, $d\xi_1^*/dU_2^A > 0$ because $d\xi_1^*/d\theta_1^* > 0$; see Proof of Proposition 1. Finally, using (13) we get

$$\frac{dU_1^E}{dU_2^A} = \frac{2\rho_1\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A}{r+\delta_1+\frac{\phi_1^2}{\sigma_1^A}\widetilde{D}_1^A} > 0,$$

which implies that $dV_1^*/dU_2^A > 0$. It remains to analyze how the late stage parameters affect U_2^A, and therefore M_1^{A*}, θ_1^*, ξ_1^*, and V_1^*. Using $U_2^E = U_2^A$ we can rewrite condition (20) from Proof of Proposition 2 as

$$U_2^A\left[r+\delta_2\right] - \frac{\phi_2^2}{\sigma_2^V}\left[D_2^V\right]^2 + \sigma_2 = 0,$$

which defines U_2^A. Implicitly differentiating U_2^E w.r.t. ϕ_2 yields

$$\frac{dU_2^A}{d\phi_2} = \frac{2\frac{\phi_2}{\sigma_2^V}\left[D_2^V\right]^2}{r+\delta_2+\frac{4}{3}\frac{\phi_2^2}{\sigma_2^V}D_2^V} > 0.$$

Thus, $dM_1^{A*}/d\phi_2$, $d\theta_1^*/d\phi_2$, $d\xi_1^*/d\phi_2$, $dV_1^*/d\phi_2 > 0$. Likewise,

$$\frac{dU_2^A}{d\sigma_2} = -\frac{1}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V} D_2^V} < 0,$$

which implies $dM_1^{A*}/d\sigma_2$, $d\theta_1^*/d\sigma_2$, $d\xi_1^*/d\sigma_2$, $dV_1^*/d\sigma_2 < 0$. Moreover,

$$\frac{dU_2^A}{d\delta_2} = -\frac{U_2^E}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V} D_2^V} < 0,$$

so that $dM_1^{A*}/d\delta_2$, $d\theta_1^*/d\delta_2$, $d\xi_1^*/d\delta_2$, $dV_1^*/d\delta_2 < 0$. Furthermore,

$$\frac{dU_2^A}{d\sigma_2^V} = -\frac{\left[\frac{\phi_2 D_2^V}{\sigma_2^V}\right]^2}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V} D_2^V} < 0.$$

This implies that $dM_1^{A*}/d\sigma_2^V$, $d\theta_1^*/d\sigma_2^V$, $d\xi_1^*/d\sigma_2^V$, $dV_1^*/d\sigma_2^V < 0$. Moreover,

$$\frac{dU_2^E}{dl_2} = -\frac{\frac{2}{3}\frac{\phi_2^2}{\sigma_2^V} D_2^V}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V} D_2^V} < 0.$$

Thus, dM_1^{A*}/dl_2, $d\theta_1^*/dl_2$, $d\xi_1^*/dl_2$, $dV_1^*/dl_2 < 0$. Finally note that $\partial D_2^V/\partial k_2 = (\partial \pi/\partial k_2)/3 = -1/3$. Thus,

$$\frac{dU_2^A}{dk_2} = -\frac{\frac{2}{3}\frac{\phi_2^2}{\sigma_2^V} D_2^V}{r + \delta_2 + \frac{4}{3}\frac{\phi_2^2}{\sigma_2^V} D_2^V} < 0,$$

so that dM_1^{A*}/dk_2, $d\theta_1^*/dk_2$, $d\xi_1^*/dk_2$, $dV_1^*/dk_2 < 0$. □

Derivation of Deal Values and Equity Shares – Hold-up of Angels.

The new coalition values are given by $CV_{EAV} = \pi$, $CV_{EA} = U_2^E + U_2^A$, $CV_{EV} = \lambda_A \pi$, and $CV_{AV} = CV_E = CV_A = CV_V = 0$. Using the general deal values (16), (17), and (18), we get

$$\begin{aligned} D_2^E &= \frac{1}{6}\left[2 + \lambda_A\right]\pi + \frac{1}{6}\left[U_2^E + U_2^A\right] \\ D_2^A &= \frac{1}{3}\left[1 - \lambda_A\right]\pi + \frac{1}{6}\left[U_2^E + U_2^A\right] \\ D_2^V &= \frac{1}{6}\left[2 + \lambda_A\right]\pi - \frac{1}{3}\left[U_2^E + U_2^A\right] \end{aligned}$$

The new equilibrium equity share for entrepreneurs, β^{E*}, ensures that their actual net payoff equals their deal value from the bargaining game: $\beta^{E*}y_2 - l_2 = D_2^E$. Solving this for β^{E*} yields

$$\beta^{E*} = \frac{l_2 + D_2^E}{y_2} = \frac{1}{6y_2}\left[6l_2 + (2+\lambda_A)\pi + U_2^E + U_2^A\right].$$

Likewise we get

$$\begin{aligned}\beta^{A*} &= \frac{D_2^A}{y_2} = \frac{1}{6y_2}\left[2(1-\lambda_A)\pi + U_2^E + U_2^A\right] \\ \beta^{V*} &= \frac{k_2 + D_2^V}{y_2} = \frac{1}{6y_2}\left[6k_2 + (2+\lambda_A)\pi - 2\left(U_2^E + U_2^A\right)\right].\end{aligned}$$

Proof of Proposition 5.

We first show that $dU_2^A/d\lambda_A < 0$. Note that $D_2^A \neq D_2^E$ for $\lambda_A > 0$, and recall that $q_2^E = \phi_2\left[M_2^{V*}/M_2^{E*}\right]^{0.5} = \phi_2^2 D_2^V/\sigma_2^V$. Thus, using (6) we define

$$\begin{aligned}F &\equiv U_2^E(r+\delta_2) + \sigma - a_2 D_2^V\left[D_2^E - U_2^E\right] = 0 \\ G &\equiv U_2^A(r+\delta_2) + \sigma - a_2 D_2^V\left[D_2^A - U_2^A\right] = 0,\end{aligned}$$

where $a_2 = \phi_2^2/\sigma_2^V$. Using Cramer's rule we get

$$\frac{dU_2^A}{d\lambda_A} = \frac{\begin{vmatrix} -\frac{\partial F}{\partial \lambda_A} & \frac{\partial F}{\partial U_2^E} \\ -\frac{\partial G}{\partial \lambda_A} & \frac{\partial G}{\partial U_2^E}\end{vmatrix}}{\begin{vmatrix} \frac{\partial F}{\partial U_2^A} & \frac{\partial F}{\partial U_2^E} \\ \frac{\partial G}{\partial U_2^A} & \frac{\partial G}{\partial U_2^E}\end{vmatrix}} = \frac{-\frac{\partial F}{\partial \lambda_A}\frac{\partial G}{\partial U_2^E} + \frac{\partial G}{\partial \lambda_A}\frac{\partial F}{\partial U_2^E}}{\frac{\partial F}{\partial U_2^A}\frac{\partial G}{\partial U_2^E} - \frac{\partial G}{\partial U_2^A}\frac{\partial F}{\partial U_2^E}}.$$

The denominator is negative if

$$\frac{\partial F}{\partial U_2^A}\frac{\partial G}{\partial U_2^E} < \frac{\partial G}{\partial U_2^A}\frac{\partial F}{\partial U_2^E},$$

which is equivalent to

$$\begin{aligned}&\frac{1}{6}a_2\left[2\left[D_2^E - U_2^E\right] - D_2^V\right]\frac{1}{6}a_2\left[2\left[D_2^A - U_2^A\right] - D_2^V\right] \\ < &\left[r + \delta_2 + \frac{1}{6}a_2\left[2\left[D_2^A - U_2^A\right] + 5D_2^V\right]\right]\left[r + \delta_2 + \frac{1}{6}a_2\left[2\left[D_2^E - U_2^E\right] + 5D_2^V\right]\right].\end{aligned}$$

If this condition holds for $r+\delta_2=0$, then it also holds for all $r+\delta_2>0$. Setting $r+\delta_2=0$ we get

$$-2\left[D_2^E-U_2^E\right]D_2^V-2\left[D_2^A-U_2^A\right]D_2^V<10\left[D_2^A-U_2^A\right]D_2^V+10D_2^V\left[D_2^E-U_2^E\right]+24\left[D_2^V\right]^2.$$

This condition is satisfied as $D_2^E>U_2^E$ and $D_2^A>U_2^A$. Thus, the denominator of $dU_2^A/d\lambda_A$ is strictly negative. Likewise, the numerator is positive if

$$\frac{\partial G}{\partial\lambda_A}\frac{\partial F}{\partial U_2^E}>\frac{\partial F}{\partial\lambda_A}\frac{\partial G}{\partial U_2^E},$$

which is equivalent to

$$\begin{aligned}&\frac{1}{6}\pi a_2\left[\left[D_2^A-U_2^A\right]-2D_2^V\right]\left[r+\delta_2+\frac{1}{6}a_2\left[2\left[D_2^E-U_2^E\right]+5D_2^V\right]\right]\\<\;&\frac{1}{6}\pi a_2\left[\left[D_2^E-U_2^E\right]+D_2^V\right]\frac{1}{6}a_2\left[2\left[D_2^A-U_2^A\right]-D_2^V\right].\end{aligned}$$

This condition can be written as

$$\frac{2}{a_2}\left(r+\delta_2\right)\left[\left[D_2^A-U_2^A\right]-2D_2^V\right]+\left[D_2^A-U_2^A\right]D_2^V-D_2^V\left[D_2^E-U_2^E\right]<3\left[D_2^V\right]^2. \quad (23)$$

From F and G we know that

$$D_2^V\left[D_2^E-U_2^E\right]=\frac{U_2^E\left(r+\delta_2\right)+\sigma}{a_2}\quad\text{and}\quad D_2^V\left[D_2^A-U_2^A\right]=\frac{U_2^A\left(r+\delta_2\right)+\sigma}{a_2},$$

so that we can write condition (23) as follows:

$$\begin{aligned}\frac{2}{a_2}\left(r+\delta_2\right)\left[\left[D_2^A-U_2^A\right]-2D_2^V\right]+\frac{U_2^A\left(r+\delta_2\right)+\sigma}{a_2}-\frac{U_2^E\left(r+\delta_2\right)+\sigma}{a_2}&<3\left[D_2^V\right]^2\\\Leftrightarrow\quad\left(r+\delta_2\right)\underbrace{\left[2D_2^A-U_2^A-4D_2^V-U_2^E\right]}_{\equiv T}&<3\left[D_2^V\right]^2a_2.\end{aligned}$$

We now show that $T<0$. Using the definitions of D_2^A and D_2^V we can write $T<0$ as

$$\begin{aligned}\frac{2}{3}\left[1-\lambda_A\right]\pi+\frac{1}{3}\left[U_2^E+U_2^A\right]-U_2^A-\frac{2}{3}\left[2+\lambda_A\right]\pi+\frac{4}{3}\left[U_2^E+U_2^A\right]-U_2^E&<0\\\Leftrightarrow\quad U_2^E+U_2^A&<\left[1+2\lambda_A\right]\pi.\end{aligned}$$

This condition is satisfied for all $\lambda_A \geq 0$ because $\pi > U_2^E + U_2^A$ by definition. Thus, the numerator of $dU_2^A/d\lambda_A$ is strictly positive. Thus, $dU_2^A/d\lambda_A < 0$. Following along the lines of Proof of Proposition 4, one can show that dM_1^{A*}/dU_2^A, $d\theta_1^*/dU_2^A$, $d\xi_1^*/dU_2^A > 0$ even when $U_2^E \neq U_2^A$. Thus, $dM_1^{A*}/d\lambda_A$, $d\theta_1^*/d\lambda_A$, $d\xi_1^*/d\lambda_A < 0$. Moreover, it is straightforward to verify that the sign of dV_1^*/dU_2^A – and therefore the sign of $dV_1^*/d\lambda_A$ – is ambiguous, and depends on the specific parameter values.

Finally note that $\partial D_2^E/\partial\lambda_A = \pi/6 < \left|\partial D_2^A/\partial\lambda_A\right| = \pi/3$. Thus, $d\left[U_2^E + U_2^A\right]/d\lambda_A < 0$, which implies that $dD_2^V/d\lambda_A > 0$. Using the definitions of θ_2^* and V_2^*, we can then infer that $d\theta_2^*/d\lambda_A > 0$ and $dV_2^*/d\lambda_A < 0$. Moreover, recall from Proof of Proposition 2 that $d\xi_2^*/d\theta_2^* > 0$. Thus, $d\xi_2^*/d\lambda_A > 0$. However, the effect on M_2^{V*} is ambiguous because $dx_1^*/d\lambda_A < 0$ (which follows from $d\theta_1^*/d\lambda_A < 0$), while $d\theta_2^*/d\lambda_A > 0$. □

Proof of Proposition 6.

Note that entrepreneurs and angels always prefer searching for a VC to an early exit if $U_2^E + U_2^A > \widetilde{y}_1$. Thus, for the trilateral bargaining game, the coalition value for the entrepreneur-angel sub-coalition is still given by $CV_{EA} = U_2^E + U_2^A$. Because they cannot sell the venture unilaterally, the coalition values for the entrepreneur sub-coalition (CV_E) and the angel sub-coalition (CV_A) are still $CV_E = CV_A = 0$. Thus, the late stage deal values do not change, and more importantly, do not depend on $\widetilde{y}_1$. Moreover, $U_2^E = U_2^A$ in equilibrium. As shown in Proof of Proposition 4, $dU_2^E/d\phi_2 > 0$, and $dU_2^E/d\sigma_2$, $dU_2^E/d\delta_2$, $dU_2^E/d\sigma_2^V$, dU_2^E/dl_2, $dU_2^E/dk_2 < 0$. Consequently, $d\widehat{y}_1/d\phi_2 > 0$, and $d\widehat{y}_1/d\sigma_2$, $d\widehat{y}_1/d\delta_2$, $d\widehat{y}_1/d\sigma_2^V$, $d\widehat{y}_1/dl_2$, $d\widehat{y}_1/dk_2 < 0$. □

Proof of Proposition 7.

Recall from Proof of Proposition 5 that $d\left[U_2^E + U_2^A\right]/d\lambda_A < 0$. Thus, $d\widehat{y}_1/d\lambda_A < 0$. □

Proof of Proposition 8.

Using $l_1 = \Psi_1 - k_1$ we can write the deal values as follows:

$$
\begin{aligned}
\widetilde{D}_1^E &= \frac{1}{2}\left[\rho_1\left(U_2^E + U_2^A\right) + (1-\rho_1)y_1 - \Psi_1 + U_1^E\right] \\
\widetilde{D}_1^A &= \frac{1}{2}\left[\rho_1\left(U_2^E + U_2^A\right) + (1-\rho_1)y_1 - \Psi_1 - U_1^E\right].
\end{aligned}
$$

Using (13) it is then easy to see that $dU_1^E/dk_1 = 0$ for $\alpha^* \in [0,1]$. Moreover, $d\widetilde{D}_1^A/dk_1 = 0$ implies that $d\theta_1^*/dk_1 = 0$, and therefore, $dM_1^{A*}/dk_1 = d\xi_1^*/dk_1 = 0$. It remains to show that $dV_1^*/dk_1 > 0$. We get

$$\frac{dV_1^*}{dk_1} = \frac{\overbrace{\alpha^* - k_1 \frac{d\alpha^*}{dk_1}}^{\equiv T}}{[\alpha^*]^2},$$

where

$$\alpha^* = \frac{1}{2} + \frac{2k_1 - \Psi_1 - U_1^E}{2(1-\rho_1)y_1}.$$

Observe that $T \geq 0$ for $k_1 \to 0$, so that $dV_1^*/dk_1 \geq 0$ for $k_1 \to 0$. To prove that $dV_1^*/dk_1 > 0$ for all $k_1 > 0$ it is therefore sufficient to show that $dT/dk_1 > 0$:

$$\frac{dN}{dk_1} = \frac{d\alpha^*}{dk_1} - \left(\frac{d\alpha^*}{dk_1} + k_1 \frac{d^2\alpha^*}{dk_1^2} \right) = -k_1 \frac{d^2\alpha^*}{dk_1^2}.$$

Next we derive the sign of $d^2\alpha^*/dk_1^*$. We get

$$\frac{d\alpha^*}{dk_1} = \frac{1}{2(1-\rho_1)y} \left[2 - \frac{dU_1^E}{dk_1} \right] = \frac{1}{2(1-\rho_1)y} \left[2 + \frac{1}{(r+\delta_1)\left[a_1 \widetilde{D}_1^A\right]^{-1} + 1} \right],$$

where $a_1 = \phi_1^2/\sigma_1^A$. Thus,

$$\frac{d^2\alpha^*}{dk_1^2} = \frac{1}{2(1-\rho_1)y_1} \frac{-\frac{1}{2}a_1 (r+\delta_1) \left[a_1 \widetilde{D}_1^A\right]^{-2} \left[1 + \frac{dU_1^E}{dk_1}\right]}{\left[(r+\delta_1)\left[a_1 \widetilde{D}_1^A\right]^{-1} + 1\right]^2},$$

where

$$1 + \frac{dU_1^E}{dk_1} = 1 - \frac{a_1 \widetilde{D}_1^A}{r + \delta_1 + a_1 \widetilde{D}_1^A} = \frac{r+\delta_1}{r + \delta_1 + a_1 \widetilde{D}_1^A} > 0.$$

Consequently, $d^2\alpha^*/dk_1^2 < 0$. This implies that $dT/dk_1 > 0$, and therefore $dV_1^*/dk_1 > 0$. □

Proof of Proposition 9.

Suppose $k_1 > \widehat{k}_1(1)$, so that $\alpha^* = 1$. The actual deal values are then given by

$$D_1^E(k_1|k_1 > \widehat{k}_1(1)) = \rho_1 U_2^E - l_1$$

$$D_1^A(k_1|k_1 > \widehat{k}_1(1)) = \rho_1 U_2^A + (1-\rho_1)y_1 - k_1.$$

It is then easy to see that $D_1^A(k_1|k_1 > \widehat{k}_1(1)) < \widetilde{D}_2^E$. Using the definition of θ_1^* we can see that this implies a lower equilibrium degree of competition. Moreover, using the definitions of M_1^{A*} and ξ_1^* we can then infer that the angel market is smaller, and has a lower deal rate, compared to the unconstrained case.

Now suppose $k_1 < \widehat{k}_1(0)$, so that $\alpha^* = 0$. The actual deal values are then given by

$$\begin{aligned} D_1^E(k_1|k_1 < \widehat{k}_1(0)) &= \rho_1 U_2^E + (1-\rho_1)y_1 - l_1 \\ D_1^A(k_1|k_1 < \widehat{k}_1(0)) &= \rho_1 U_2^A - k_1. \end{aligned}$$

We can immediately see that $D_1^A(k_1|k_1 < \widehat{k}_1(0)) < \widetilde{D}_2^E$. Using again the definitions of θ_1^*, M_1^{A*} and ξ_1^*, we can infer that the angel market is then larger and more competitive compared to the unconstrained case, and has a higher deal rate. □

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